

# VANISHING OBSTRUCTIONS TO SLICENESS OF WEAVING LINKS

by

**Parker Smiley**

An Abstract

presented in partial fulfillment  
of the requirements for the degree of  
Master of Science  
in the Department of Mathematics, Actuarial Sciences, and Statistics  
University of Central Missouri

May, 2026

# ABSTRACT

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A weaving link  $W(p, q)$  is the closure of the alternating plait braid on  $p$  strands with  $q$  repetitions. With braid generators  $\sigma_i$ , we write

$$W(p, q) = \left\langle \left( \sigma_1 \sigma_2^{-1} \sigma_3 \cdots \sigma_{p-1}^{(-1)^p} \right)^q \right\rangle$$

We are interested in what conditions and obstructions arise that determine whether a weaving knot is *slice*. A knot  $K$  in the 3-sphere is called *slice* if it bounds a smooth 2-dimensional disk properly embedded in the 4-ball. In pursuit of solving problems related to the sliceness of weaving knots, we first explicitly compute the *Seifert form* (by extension the *Seifert matrix*) for all weaving links and explore known invariants of weaving links and their computations. The Seifert matrix is used to compute the signature of  $W(p, q)$  explicitly, and further to reduce the computational complexity of the Alexander polynomial over traditional methods. We then show how these classical invariants fail to obstruct sliceness in  $W(p, q)$  when  $p \geq 3$  is odd and when  $q \neq 2$ . We use symmetric properties of weaving links to prove that many of our stronger invariants in knot theory fail obstruct sliceness for weaving links. We conclude by proposing potential alternative invariants that may be used as sliceness obstructions.

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## ACKNOWLEDGMENTS

I would be foolish to not thank my thesis advisor Dr. Nick Meyer for his direction and encouragement, as well as the inspiration to take on this project. He has been supportive of my interest in topology since he began his position at UCM in 2024.

I would like to thank the Department of Mathematics, Actuarial Science and Statistics at the University of Central Missouri. Thank you to the numerous faculty that have supported my coursework and program planning. To Dr. Heider, thank you for supporting my independent studies in algebra during my graduate program. To Dr. Fonseca, thank you for your advice and encouragement about applying to Ph.D. programs.

To Ryan and the *boys*, thank you for listening to my excited rambling about obscure math facts. You guys have kept me grounded and kept me accountable for taking breaks. Even if this is the only part of this paper you read, know that you have already heard 90% of it in our conversations.

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# 1 INTRODUCTION

Weaving links are braid closures of alternating plait braid words with  $p$  strands and  $q$  repetitions. These links, having a simple braid word construction, are known to evade many preexisting obstructions to sliceness. What began as a project to find a working obstruction for sliceness has turned into an exploration of the vanishing obstructions, as well as the properties of link symmetry that cause these obstructions to vanish. We begin with an exposition of weaving links and some known properties of these links. We then explore some of the commonly used invariants of knot theory and compute them for weaving links. Particularly, we demonstrate how strongly negative amphicheirality causes many of the strongest currently used tools to vanish in weaving links. We demonstrate proofs for our main theorems, and propose future research directions and open questions for these links.

## 1.1 Knots and Links

By a knot we shall always mean an embedding of the circle  $S^1$  into the 3-sphere  $S^3$ . Similarly, a link of  $n$  components is a disjoint union of  $n$  embedded circles that may be tangled or knotted together in some fashion. We say that two links are equivalent if one can be deformed into the other by an ambient isotopy, a distortion of the link into the other that does not break the link strands at any point. We call any knot equivalent to a knot with zero crossings the *unknot* or *trivial knot*, as depicted in Figure 1.1.

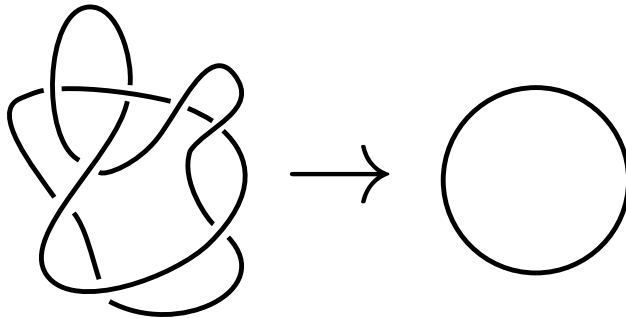


Figure 1.1: A tangled knot which is ambient isotopic to the unknot

We can construct links through the construction of *braids*. A *braid on  $n$  strands* is a set of  $n$  oriented curves in  $n$  different positions traversing from top to bottom, and crossing over one another by a set of  $n - 1$  generators. Labeling the strands  $s_1, s_2, \dots, s_n$  from left to right, the process of each crossing generator  $\sigma_i$  is to cross strand  $s_i$  over strand  $s_{i+1}$ , so that  $s_i$  is traversing in position  $i + 1$  and  $s_{i+1}$  in position  $i$ . See the braid word  $b = \sigma_1 \sigma_1 \sigma_3^{-1} \sigma_2 \sigma_2$  in Figure 1.2 for example. In the same way,  $\sigma_i^{-1}$  crosses  $s_i$  under strand  $s_{i+1}$  with the same change in position. We may then notate a braid  $b$  through its *braid word*, which is the

product of its crossing generators  $\sigma_i$ . The *closure* of a braid  $b$  is a link formed by connecting each bottom strand  $s_i$  in position  $j$  to the top of strand  $s_j$ , and is denoted  $\langle b \rangle$ .

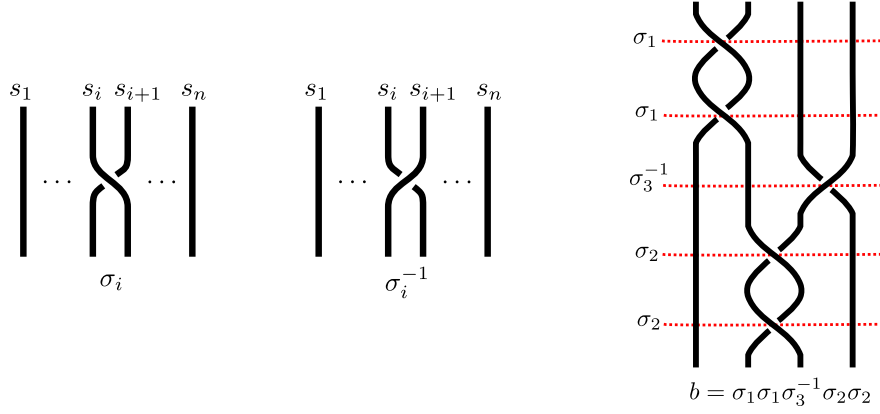


Figure 1.2: The crossing generator  $\sigma_i$  and its inverse in a braid word

## 1.2 Construction of Weaving Links

We now define weaving links in terms of braid closures. Let  $p, q \in \mathbb{N}$ . The  $(p, q)$ -weaving link on  $p$  strands with  $q$  repetitions is the link formed by taking the closure of the  $p$  strand braid  $(\sigma_1 \sigma_2^{-1} \dots \sigma_{p-1}^{\pm 1})^q$ , and is denoted  $W(p, q)$ . When  $\gcd(p, q) = 1$ , then  $W(p, q)$  is a 1-component link that we call a weaving *knot* on  $p$  strands with  $q$  repetitions. See Figures 1.3 and 1.4 for examples of the standard diagrams of  $W(3, 2)$  and  $W(4, 3)$  respectively.

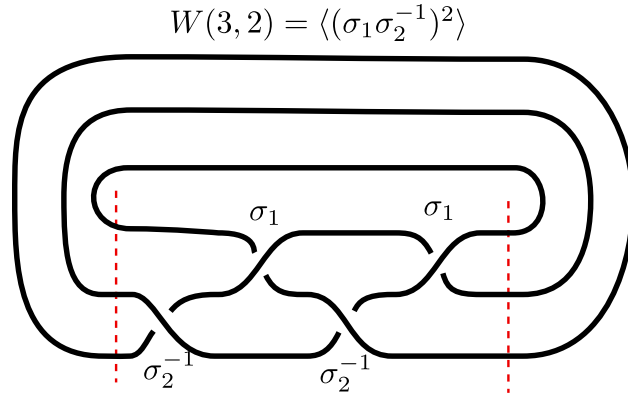
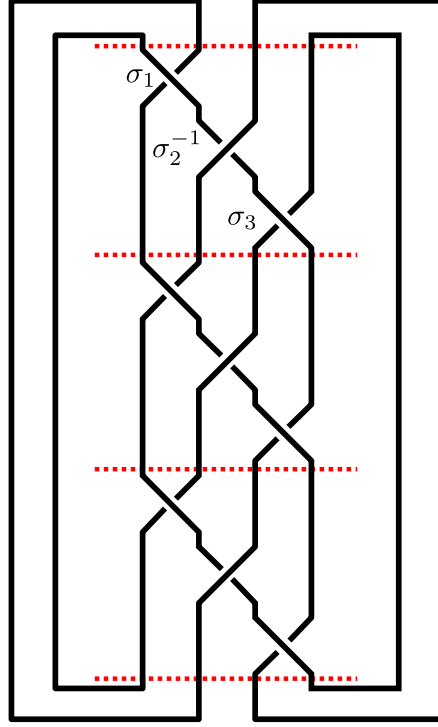


Figure 1.3: Construction of  $W(3, 2)$  as a braid closure

Note that  $W(1, q)$  and  $W(p, 1)$  are equivalent to the unknot. Further,  $W(2, q)$  is simply the alternating torus link  $T(2, q)^1$ . Torus links are well-studied, so we avoid discussing them here. We restrict our view to weaving links with  $p \geq 3$  and  $q \geq 2$  for the remainder of this work.

<sup>1</sup>The torus link  $T(p, q)$  embeds on the surface of an unknotted torus in  $S^3$ , winding  $p$  times on the meridional direction of the torus and  $q$  times around the longitudinal direction. See [Rol03] pg. 53-55



$$W(4, 3) = \langle (\sigma_1 \sigma_2^{-1} \sigma_3)^3 \rangle$$

Figure 1.4: Construction of  $W(4, 3)$  as a braid closure

## 2 PRELIMINARIES

We now explore and expound upon known properties of weaving links. We begin by discussing the symmetric properties of weaving links. We define the Seifert surface of a link, and provide two particular variations of Seifert surfaces for weaving links. Additionally, we demonstrate how a Seifert surface for a weaving link may be realized as an open book decomposition of the link's exterior, demonstrating that weaving links are *fibred* links. We demonstrate how the Seifert surface of a link may be used to form a Seifert matrix, which is used to compute many classical knot invariants. We provide a fast formula to compute the Seifert matrix of a weaving link as a block-tridiagonal matrix, which we will use to support our primary results. We define what an invariant of a knot is and provide two brief examples. Finally, we define slice knots in terms of disk boundaries and knot concordance.

### 2.1 Symmetries

Let  $K$  be an oriented knot in  $S^3$ . By changing the orientation of  $K$  we obtain the *reverse* of  $K$ , denoted  $-K$ . In the same way, by changing the crossings of  $K$  we obtain the *reflection* of  $K$ , denoted  $\overline{K}$ . With these

three variations of a knot, we describe the primary types of symmetries that may interest us.

**Definition 2.1.** Let  $K$  be an oriented knot in  $S^3$ .

1. We say  $K$  is **invertible** if  $K$  is isotopic to  $-K$ .
2. We say  $K$  is **positively amphicheiral** if  $K$  is isotopic to  $\overline{K}$ .
3. We say that  $K$  is **negatively amphicheiral** if  $K$  is isotopic to  $-\overline{K}$ .

If the isotopy used in any of these definitions is found to be an involution (i.e.  $f^2 = \text{id}$ ), then the corresponding property is called **strong**.

$W(p, q)$  is strongly invertible by a rotation of  $\pi$  radians on the horizontal axis (for our standard diagram). Further, if  $p$  is odd, then  $W(p, q)$  is strongly negatively amphicheiral, and strongly positively amphicheiral if both  $p$  and  $q$  are odd.

## 2.2 Seifert Surfaces and Fiberedness

We can treat any link as the boundary of some compact, connected surface. As an example, consider the trefoil  $3_1$  (equiv.  $W(2, 3)$ ) as the boundary of a Möbius strip. However, we would like to restrict our view to those surfaces which are connected, compact, and orientable.

**Definition 2.2** ([Lic97], Def. 2.1). Let  $L \subseteq S^3$  be an oriented link. A Seifert surface  $F \subseteq S^3$  for  $L$  is a connected, compact, oriented surface that has  $L$  as its oriented boundary.

In 1934, Seifert [Sei35] showed that any oriented link in  $S^3$  has a Seifert surface. Methods for finding Seifert surfaces exist, but we will use a consistent method that works for any weaving links encountered. The Seifert surface for a weaving link consists of the  $p$  Seifert circles (copies of  $D^2$ ) bounded by each strand when the crossings are eliminated. At each crossing, we glue a positive (resp. negative) half-twist (See Figure 2.2) between the two circles whose boundary is the crossing generated by  $\sigma_i$  ( $\sigma_i^{-1}$ ). The resulting surface may be oriented, making it a Seifert surface.

Let  $L \subseteq S^3$  be a link. Let  $\nu(L)$  be a tubular neighborhood of the link  $L$ . We say that a link  $L$  is *fibered* if the link exterior

$$E(L) = S^3 \setminus \nu(L)$$

admits a smooth projection  $\pi : E(L) \rightarrow S^1$  such that for each  $\theta \in S^1$ , there exists a neighborhood  $U_\theta \subseteq S^1$

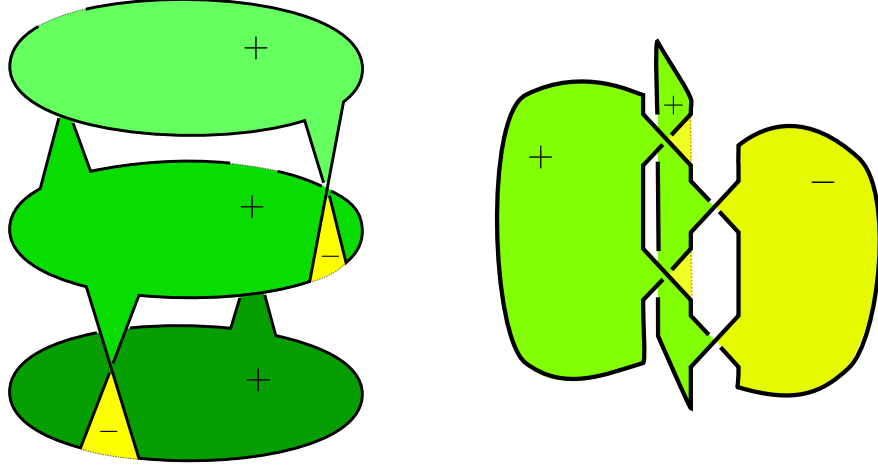


Figure 2.1: “Wedding cake” and “Book” Seifert surfaces for  $W(3,2)$

of  $\theta$  such that

$$\pi^{-1}(U_\theta) \cong F \times U_\theta,$$

where  $F$  is a Seifert surface for  $L$ . In particular, each fiber  $\pi^{-1}(\theta)$  is isotopic to a copy of  $F$ . Equivalently, we can consider the link  $L$  as the binding of an open book decomposition of the exterior of  $L$ . Then we say  $L$  is *fibred* if there exists a Seifert surface  $F$  for  $L$  so that the exterior of  $L$  is filled by copies of  $F$  in a circular arrangement, like pages attached to a binding.

We say that a braid word  $b$  is *homogeneous* if each  $\sigma_i$  occurs at least once, and for each  $i$ , the exponent of each occurrence of  $\sigma_i$  is the same. Thus, all weaving links have homogeneous braid words. By work of Stallings (Theorem 2 of [Sta78]), all braid closures with homogeneous braid words, such as weaving links, are fibred links.

### 2.3 Seifert Matrix

Let  $x$  and  $y$  be two oriented closed curves. Label each crossing  $c_i$  with either a 1, -1, or 0 based on the interactions depicted in Figure 2.2. Then the *linking number* between  $x$  and  $y$  is  $\text{lk}(x, y) = \frac{1}{2} \sum c_i$ . An example of such a computation is provided in Figure 2.3.

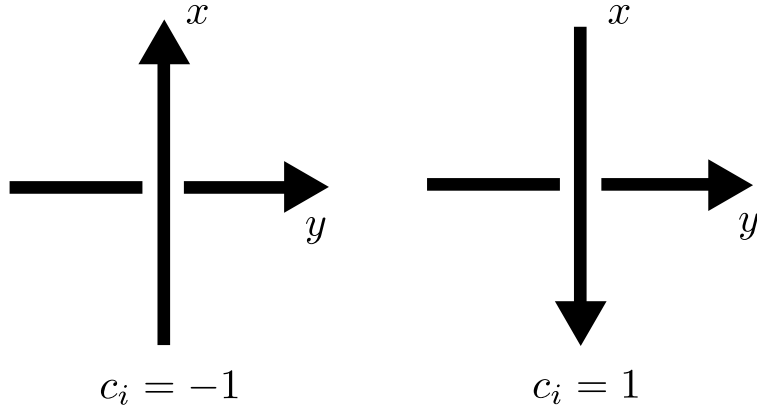


Figure 2.2: Assignment value for each crossing  $c_i$

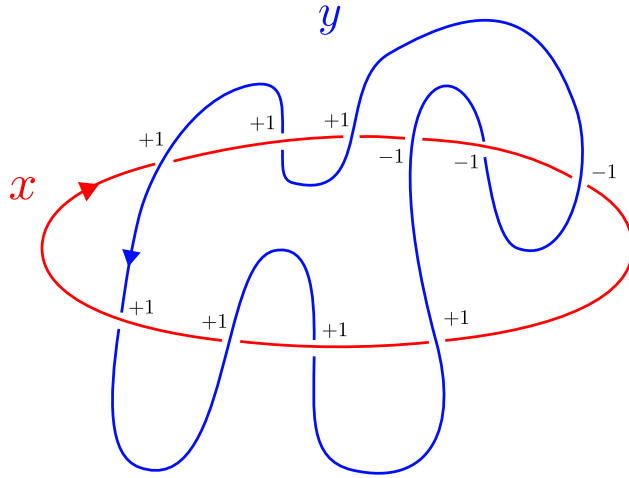


Figure 2.3: Two closed curves  $x$  and  $y$  with  $\text{lk}(x, y) = 2$

In particular, we have that for the simplest nontrivial linking of two oriented unknots the following linking numbers in Figure 2.4. It should be noted that linking number is invariant under ambient isotopy, making it a useful tool for constructing knot invariants.

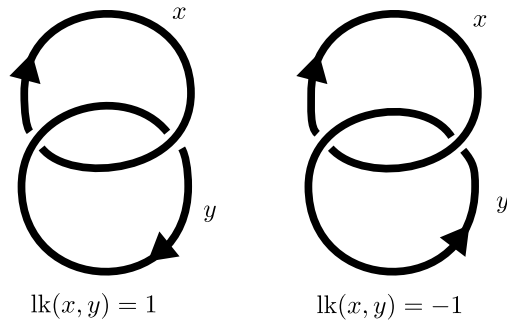


Figure 2.4: Linking number for two simple closed curves

We can now describe the construction of the Seifert form. Given a Seifert surface  $V_L$  of genus  $g$  for a

link  $L$ , we identify a basis of  $g$  generating cycles  $\{e_1, e_2, \dots, e_g\}$  for  $V_L$ . By nudging a cycle  $e_i$  slightly off the surface in the positive direction we have a loop  $e_i^+$  and we examine the linking numbers  $\text{lk}(e_i, e_j^+)$ . This linking number is called the *Seifert form*. We then construct a  $2g \times 2g$  matrix with entries  $\text{lk}(e_i, e_j^+)$ , which we call a *Seifert matrix* for  $L$ .

In a recent (2025) preprint by Blackwell et. al on spiral knots [BDM<sup>+</sup>25], a method for calculating the Seifert matrix can be found that is quite similar to ours. Spiral knots represent a generalized notion of weaving links, wherein the braid word need not be alternating. For the general reader the following approach may prove more digestible and repeatable for weaving links.

We now present the computation for a Seifert matrix for weaving links:

**Theorem 2.3.** Let  $p, q \in \mathbb{Z}$  with  $p \geq 3, q \geq 2$ . Let  $A$  and  $B$  be  $(p-1) \times (p-1)$  matrices defined by

$$(A)_{ij} = \begin{cases} (-1)^{i+1}, & i = j \\ -1, & j = i + 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (B)_{ij} = \begin{cases} 1, & (i = j) \text{ and } 2|i \\ 0 & \text{otherwise} \end{cases}$$

Let  $C = -A - B$ . The  $(q-1) \times (q-1)$  block-tridiagonal matrix  $V_{p,q}$  defined by

$$V_{p,q} = \begin{bmatrix} A & B & 0 & \dots & 0 \\ C & A & B & \ddots & \vdots \\ 0 & C & A & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & B \\ 0 & \dots & 0 & C & A \end{bmatrix}.$$

is a Seifert matrix for  $W(p, q)$ .

We will use this Seifert matrix to compute various invariants for weaving links.

## 2.4 Invariants

An *invariant* of a knot is a quantifiable property defined for all knots such that the property's value is the same for two equivalent knots. One example of this is through diagrammatic invariants. Two knots  $K$  and  $J$  are equivalent if they may be expressed by the same knot diagram.

We present one example of a diagrammatic knot invariant and compute it for weaving links. The *crossing number*  $c(K)$  of a knot  $K$  is defined to be the smallest number of crossings of any diagram of  $K$ . Note that

crossing number is only an invariant when the diagram is minimal.

We may compute the crossing number for weaving links. Murasugi provides the following theorem from his solution to the first of the Tait conjectures [Mur87].

**Theorem 2.4.** Any reduced diagram of an alternating link has the fewest possible crossings.

The standard diagram (Figure 1.3) for weaving links is a reduced alternating diagram. Thus,

**Corollary 2.5.** The minimal number of crossings for the weaving link  $W(p, q)$  is given by the  $p - 1$  crossing generators of the braid word, which are repeated  $q$  times. That is,

$$c(W(p, q)) = (p - 1)q.$$

Knot invariants are tools that allow us to discern between two different knots without relying on visual examination (a rather tedious task for knot diagrams with many crossings). We wish to find properties and invariants of weaving knots which may be used to prove that certain weaving knots are not slice. Such properties would be called *obstructions* to sliceness.

## 2.5 Genus

The 3-genus is an example of an invariant of the link itself, unlike crossing number, which is a *diagrammatic* invariant.

**Definition 2.6.** The *3-genus* of a link  $L$  is the minimum genus of all Seifert surfaces for  $L$ , and is denoted  $g_3(L)$ .

Up to ambient isotopy this is a link invariant. Crowell [Cro59] demonstrated that if  $L$  is an alternating link, then the genus of any Seifert surface  $F$  that is constructed with respect to any nontrivial, alternating diagram of  $L$  is minimal, so that  $g(F) = g_3(L)$ . Rolfsen [Rol03] provides the formula for the genus of a Seifert surface  $F$  for a link  $L$  as an exercise:

$$g(F) = 1 - \frac{s + n - c}{2}$$

where  $n$  is the number of link components,  $c$  is the number of crossings and  $s$  is the number of Seifert circles in the diagram. In Figure 2.1, we presented the open book description for the fiber surface of  $W(p, q)$ . We

will use this construction for a Seifert surface  $F$  of  $W(p, q)$ . This surface is constructed on a nontrivial, alternating diagram of  $W(p, q)$ , so the genus of  $F$  is equal to  $g_3(W(p, q))$ . Moreover, values of  $n$ ,  $c$  and  $s$  may be found for  $W(p, q)$ :

$$n = \gcd(p, q), \quad c = q(p - 1) \quad \text{and} \quad s = p.$$

Therefore, the genus for the fiber surface  $F$  of  $W(p, q)$  is provided by

$$g(F) = 1 - \frac{p + \gcd(p, q) - q(p - 1)}{2} = \frac{(p - 1)(q - 1) - \gcd(p, q) + 1}{2}.$$

Thus, with Crowell's result, we have

**Theorem 2.7.** The 3-genus of the weaving link  $W(p, q)$  is given by

$$g_3(W(p, q)) = \frac{(p - 1)(q - 1) - \gcd(p, q) + 1}{2}.$$

## 2.6 Slice Knots

**Definition 2.8.** Let  $K$  be a knot in  $S^3$ , and treat  $S^3$  as the boundary of the 4-ball. We say that  $K$  is *smoothly slice* if  $K$  bounds a smoothly embedded disk in the boundary of the 4-ball.

For the general reader, a knot that is “slice” is akin to being unknotted in  $B^4$  despite being a (potentially) non-trivial knot in  $S^3$ . The unknot is certainly an example of a slice knot. It is an engaging exercise to show that for any oriented knot  $K$ ,  $K \# -\overline{K}$  is slice. Knots of this construction are examples of what are known as a *ribbon* knots. It is a well-known fact that every ribbon knot is a slice knot. It is a long standing conjecture (first proposed by Ralph Fox [Fox62]) that every slice knot is ribbon.

Another way to define a slice knot is through *concordance*.

**Definition 2.9.** We say two knots  $K$  and  $J$  are *concordant* if there exists a smooth embedding of a cylinder  $S^1 \times [0, 1]$  in  $S^3 \times [0, 1]$  so that  $\partial(S^1 \times \{0\}) \cong K$  and  $\partial(S^1 \times \{1\}) \cong J$ .

We call the embedding  $S^1 \times [0, 1] \hookrightarrow S^3 \times [0, 1]$  satisfying these boundary conditions a *concordance* mapping. It is well known that two knots  $K$  and  $J$  are concordant if and only if  $K \# J$  is a slice knot, and further that a knot  $K$  is slice if and only if it is concordant to the unknot. See Figure 2.5

As in the last section, we may define the *4-genus* (equiv. *slice genus*) of a knot as follows:

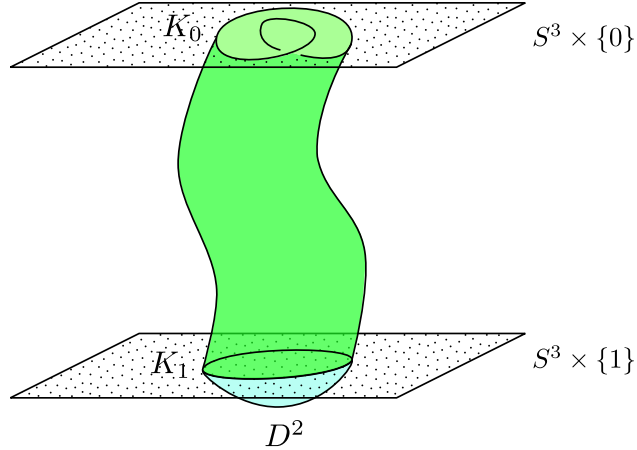


Figure 2.5: A knot  $K_0$  that is concordant to the unknot,  $K_1$ , which is slice.

**Definition 2.10.** The  $4$ -genus of a knot  $K$  is the minimum genus of any smooth, properly embedded orientable surface in  $B^4$  with  $K$  as its boundary, denoted  $g_4(K)$ .

Clearly,  $g_4(K) = 0$  if and only if  $K$  is a smoothly slice knot. Thus, many strategies for proving sliceness involve finding upper bounds on the 4-genus, and showing that these upper bounds vanish. On the other hand, obstructing sliceness involves proving that some lower bound on the 4-genus are nonzero.

Let  $W(p, q)$  be a weaving knot with  $\gcd(p, q) = 1$  and  $p \geq 3$  odd. We call a knot  $\mathbb{Q}$ -slice (equiv. rationally slice) if it bounds a smooth disk in a rational homology 4-ball. It should be noted that every slice knot is  $\mathbb{Q}$ -slice, but not every  $\mathbb{Q}$ -slice knot is slice ( $4_1$  is an example of this). Kawauchi [Kaw09] showed that every strongly negatively amphicheiral knot (SNAK) is  $\mathbb{Q}$ -slice.

At present, a few weaving knots have been proven to be smoothly slice. The knots  $W(3, 5) = 10_{123}$ ,  $W(5, 3) = 12a_{1019}$ , and  $W(7, 2) = 12a_{477}$  have been shown to be smoothly slice (verifiable on KnotInfo [LM26]), whose diagrams are provided in Figure 2.6. We wish to find invariants of weaving knots that can be used as obstructions to sliceness.

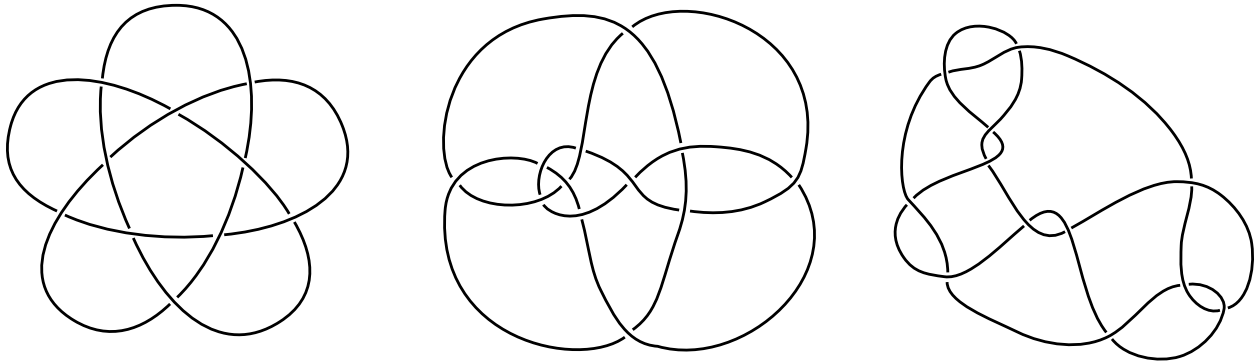


Figure 2.6: From left to right:  $W(3, 5)$ ,  $W(5, 3)$  and  $W(7, 2)$

### 3 PRIMARY RESULTS

We compute explicitly the classical signature of weaving links, and demonstrate that this invariant vanishes for odd  $p$  by a symmetry argument and a linear algebra argument. We also show how the more powerful Tristram-Levine signature vanishes in the same way. We explore the Alexander module, and its more commonly related invariant known as the Alexander polynomial. We provide a formula for the Alexander polynomial of weaving links using a formula for the determinant of block-tridiagonal matrices. We demonstrate that the knot determinant successfully obstructs sliceness in all weaving links of the form  $W(p, 2)$  with  $p \geq 3$ , except for the case that  $p = 7$ . Finally, we show how the symmetric properties of weaving links cause many powerful invariants to fail at obstructing sliceness.

#### 3.1 Signature Invariants

Let  $S$  be a symmetric matrix with real entries. Let  $\beta_+(S)$  be the number of positive eigenvalues of  $S$  and  $\beta_-(S)$  the number of negative eigenvalues of  $S$ . We call the number  $\sigma(S) = \beta_+(S) - \beta_-(S)$  the *signature* of the matrix  $S$ . Further, if  $V_K$  is a Seifert matrix for a knot  $K \subseteq S^3$ , then  $V_K + V_K^T$  is a symmetric matrix and we define  $\sigma(K) = \sigma(V_K + V_K^T)$  to be the *signature of  $K$* .

Murasugi demonstrated that knot signature is a concordance invariant:

**Theorem 3.1** (Murasugi, [Mur65]). Let  $K$  and  $J$  be knots in  $S^3$ . If  $K$  and  $J$  are concordant knots, then

$$\sigma(K) = \sigma(J).$$

Note that the unknot must have signature zero, as its Seifert surface has no genus. If a knot is smoothly slice, it must be concordant to the unknot. Therefore:

**Corollary 3.2** (Murasugi, [Mur65]). If  $K$  is smoothly slice, then  $\sigma(K) = 0$ .

Thus, knot signature proves to be a potential obstruction to sliceness.

One property of signature is that for an oriented knot  $K$ , we have  $\sigma(\overline{K}) = -\sigma(K)$ . Let  $p \geq 3$  be odd, and  $K = W(p, q)$ . Then  $K$  is strongly negatively amphicheiral, and its signature may be computed as follows:

$$\sigma(\overline{K}) = -\sigma(K) \implies \sigma(\overline{K}) + \sigma(K) = 0 \implies \sigma(K) = 0$$

Therefore, signature fails to be an obstruction for sliceness for weaving knots with odd  $p$ .

We nonetheless compute explicitly the signature of  $W(p, q)$ , and observe that it has a somewhat satisfying result for the case that  $p$  is even. The following result is not original, but lacking access to the original source material we have elected to reprove this using theorems of inertia demonstrated by Da Fonseca [dF04]:

**Theorem 3.3.** Let  $p, q \in \mathbb{Z}$  with  $p \geq 3, q \geq 2$ .

$$\sigma(W(p, q)) = \begin{cases} 0 & p \text{ is odd} \\ q - 1 & p \text{ is even} \end{cases}$$

*Proof.* Let  $W(p, q)$  be a weaving link with  $p \geq 3, q \geq 2$ , and  $V_{p,q}$  the Seifert matrix for  $W(p, q)$  as defined in Theorem 2.3. Let  $S = V_{p,q} + V_{p,q}^T$ . Then

$$S = \begin{bmatrix} A + A^T & B + C^T & 0 & \dots & 0 \\ C + B^T & A + A^T & B + C^T & \ddots & \vdots \\ 0 & C + B^T & A + A^T & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & B + C^T \\ 0 & \dots & 0 & C + B^T & A + A^T \end{bmatrix} = \begin{bmatrix} A + A^T & -A^T & 0 & \dots & 0 \\ -A & A + A^T & -A^T & \ddots & \vdots \\ 0 & -A & A + A^T & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -A^T \\ 0 & \dots & 0 & -A & A + A^T \end{bmatrix}$$

As a result of Da Fonseca we have  $\beta_+(S) = (q - 1) \cdot \beta_+(A + A^T)$  and  $\beta_-(S) = (q - 1) \cdot \beta_-(A + A^T)$  (See Theorem 2.1 of [dF04]). Let  $M = A + A^T$ . Then

$$M = A + A^T = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & -2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & -1 & (-1)^p \cdot 2 \end{pmatrix}$$

We note the symmetry and the alternating signs of the diagonal entries. Da Fonseca showed that for a symmetric matrix with alternating signs on diagonal entries, that  $\beta_+(M) = \beta_-(M) = \frac{(p-1)-1}{2} = \frac{p-2}{2}$  if  $p-1$  is even, and  $\beta_+(M) = \frac{(p-1)+1}{2}$  and  $\beta_-(M) = \frac{(p-1)-1}{2}$  if  $p-1$  is odd. We have the following:

- If  $p$  is odd, we have  $\sigma(S) = (q - 1) \frac{p-2}{2} - (q - 1) \frac{p-2}{2} = 0$ .
- If  $p$  is even, then  $\sigma(S) = (q - 1) \frac{(p-1)+1}{2} - (q - 1) \frac{(p-1)-1}{2} = q - 1$ .

□

A related, yet stronger signature-type invariant to be considered is the *Tristram-Levine* signature. Let  $\omega \in S^1 \setminus \{1\}$  and let  $V_L$  be the Seifert matrix for a link  $L$ . The Tristram-Levine Signature is a function  $\sigma_L : S^1 \setminus \{1\} \rightarrow \mathbb{Z}$  defined by

$$\sigma_L(\omega) = \sigma((1 - \omega)V_L + (1 - \bar{\omega})V_L^T)$$

Note that  $\omega_L(-1) = \sigma(L)$ , as multiplying by a positive scalar does not affect signature. The Tristram-Levine signature *can* provide more information about a knot's properties that we will not cover, except that it is able to detect non-sliceness in knots that “traditional” knot signature fails to detect.

However, much like signature, Tristram-Levine signature is a symmetric invariant, i.e.  $\sigma_L = \sigma_{\bar{L}}$ . Therefore, Tristram-Levine signature, along with many other signature-type invariants, vanish for strongly negative amphicheiral knots. We will explore this later.

### 3.2 Alexander Polynomial

Many knot invariants are constructed by observing the knot complement. Letting  $\nu(K)$  be a tubular neighborhood of a knot  $K \subseteq S^3$ , we define the knot complement (equiv. *knot exterior*) to be

$$E(K) = S^3 \setminus \nu(K).$$

$E(K)$  has an infinite cyclic cover  $\widetilde{E(K)}$ , which may be constructed by cutting the exterior along a thickened Seifert surface  $\Sigma_K$  of  $K$  and gluing infinitely many copies of the exterior end-to-end along their boundaries. We can choose  $\tau$  to be a generator for the deck transformation group. Then there is an isomorphism

$$\tau_* : H_1(\widetilde{E(K)}; \mathbb{Z}) \rightarrow H_1(\widetilde{E(K)}; \mathbb{Z})$$

induced by  $\tau$ , granting the homology group the structure of a  $\mathbb{Z}[t, t^{-1}]$ -module, which we call the *Alexander module*. This module is a knot invariant.

More commonly used (and what we will be constructing) is an invariant called the *Alexander polynomial*. Given the Alexander module of a knot  $K$ , we can build a finite presentation matrix for this module using our Seifert matrix from before. If  $V_K$  is a Seifert matrix for a knot  $K \subseteq S^3$ , then  $V_K - tV_K^T$  is a presentation matrix for the Alexander module of  $K$  (pg. 207 of [Rol03]). Taking the determinant of this resulting matrix, we have the Alexander polynomial  $\Delta_K(t)$  of  $K$ . It should be noted that the Alexander polynomial is defined up to multiplication by a unit. That is, if  $f(t)$  and  $g(t)$  are two Alexander polynomials for a knot  $K$ , then  $g(t) = \pm t^k \cdot f(t)$  for some integer  $k$ . We often write  $\Delta_K(t) \doteq f(t)$  to denote this equivalence.

A concise formula for the Alexander polynomial of a weaving link in terms of  $p$  and  $q$  has not been found

at time of writing. Most existing formulations rely on recursion and heavy determinant computations. We provide here a formulation that relies on recursion to compute the determinant of two (potentially only one) matrices.

We list some properties of the Alexander polynomial for weaving links. One classical result states that the Alexander polynomial for a fibered knot is monic.

We may also discuss the degree of the polynomial. It is well known that  $\deg(\Delta_K(t)) \leq 2g_3(K)$ . Further, Murasugi [Mur58] showed that this is actually an equality for alternating knots, such as weaving knots. More generally, Crowell [Cro59] proved that for alternating links

$$\deg(\Delta_{W(p,q)}(t)) = 2g_3(W(p,q)) + \gcd(p,q) - 1.$$

Therefore,

$$\deg(\Delta_{W(p,q)}(t)) = 2 \left( \frac{(p-1)(q-1) - \gcd(p,q) + 1}{2} \right) + \gcd(p,q) - 1$$

Thus, we have

**Theorem 3.4.** The Alexander polynomial  $\Delta_{W(p,q)}(t)$  for  $W(p,q)$  is a monic, sign-alternating Laurent polynomial with  $\mathbb{Z}$ -coefficients whose degree is given by the equation

$$\deg(\Delta_{W(p,q)}(t)) = \frac{(p-1)(q-1)}{2}.$$

We now discuss the formulation of the Alexander polynomial for weaving links. The polynomial may be written as the determinant of a finite sum of matrices  $(M^{-1}N)^i$ , where

$$M^{-1}N = \begin{cases} -t^{\hat{i}-1}, & j \leq i, j = 1 \\ (1-t)t^{\hat{i}-\hat{j}}, & j \leq i, j \text{ is even} \\ (1-t^{-1})t^{\hat{i}-\hat{j}}, & j \leq i, j \text{ is odd} \\ t^{-1}, & j = i+1, j \text{ is even} \\ 1, & j = i+1, j \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$$

where  $\hat{i} = \lfloor \frac{i}{2} \rfloor$  and  $\hat{j} = \lfloor \frac{j}{2} \rfloor$ .

Formally, we will have proven that:

**Theorem 3.5.** Let  $p, q \in \mathbb{Z}$  with  $p \geq 3, q \geq 2$ . Let  $M = B - tC^T$  and  $N = C - tB^T$ . Then up to multiplication by a unit, we have

$$\Delta_{W(p,q)}(t) \doteq \det \left( \sum_{i=0}^{q-1} (M^{-1}N)^i \right)$$

We will prove this theorem in Section 4. The matrix  $M^{-1}N$  has the nice property that it is a lower *Hesenberg* matrix, meaning it is a dense lower triangular matrix with nonzero entries on its first superdiagonal. As a result, the computation of its determinant can be done with less complexity. This has the form of a finite geometric sum.

**Corollary 3.6.** Let  $p, q \in \mathbb{Z}$  with  $p \geq 3, q \geq 2$ . Let  $M = B - tC^T$  and  $N = C - tB^T$ . Then up to multiplication by a unit, we have

$$\Delta_{W(p,q)}(t) \doteq \frac{\det(I - (M^{-1}N)^q)}{\det(I - M^{-1}N)}.$$

**Conjecture 3.7.** We may simplify the calculation of the denominator as follows:

$$\det(I - M^{-1}N) = t^{-\lfloor \frac{p}{2} \rfloor} \sum_{i=0}^{p-1} t^i \doteq \frac{t^p - 1}{t - 1}$$

Thus, up to multiplication by a unit, we have

$$\Delta_{W(p,q)}(t) \doteq \frac{t - 1}{t^p - 1} \det(I - (M^{-1}N)^q).$$

**Remark 3.1.** *It should be noted that in hindsight of this construction, it was discovered to not be original. In fact, it may be generalized to the braid representations of any link  $L$ . We direct readers to the work of Burau [Bur35], who realized that every Alexander polynomial may be constructed by*

$$\Delta(t) \doteq \frac{\det(I - \Psi)}{1 + t + \dots + t^{n-1}}$$

*where  $\Psi$  is the reduced Burau representation of the braid word of  $L$ . We also direct readers to Jones [Jon87] for a modern treatise on the subject.*

### 3.3 Knot Determinant

The determinant of a knot  $K$  is best defined by  $\det(K) = |\Delta_K(-1)|$ . It is a known result that if  $K$  is a slice knot, then  $\det(K)$  is a square integer.

Dowdall, Mattman, Meek and Solis [DMMS10] computed the determinants for all weaving links except the case that  $p \geq 5$  is odd,  $q \geq 3$  and  $\gcd(p, q) = 1$ . We focus on a specific case. For weaving links of the form  $W(p, 2)$  with  $p \geq 3$ , we have that

$$\begin{aligned} \det(K) &= |\det(V_{p,2} + V_{p,2}^T)| \\ &= \begin{vmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & -2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & -1 & 2(-1)^{p-1} \end{vmatrix} \\ &= P_p \end{aligned}$$

where  $P_0 = 0, P_1 = 1$  and  $P_k = 2P_{k-1} + P_{k-2}$  is the  $k$ -th Pell number. In 1942, Ljunggren proved that the only square Pell numbers are 0, 1, and 169<sup>2</sup>. Thus, we have the following (not necessarily new) result:

**Theorem 3.8.** The only slice weaving knot of the form  $W(p, 2)$  with  $p \geq 3$  is  $W(7, 2)$ .

### 3.4 Stronger Invariants of Alternating SNAKs that Vanish

All the invariants we have explored so far fall under the category of “classical” knot invariants. Many stronger invariants are built off of classical invariants. Some of the most powerful knot tools we currently use to study obstructions to sliceness lie in the fields of Heegaard-Floer homology, Khovanov homology, and other distinct fields. Developing invariants from these areas requires significant additional background and lies beyond the scope of this thesis. Regardless, we explore how the symmetric properties of weaving links cause many stronger tools to vanish, hopefully enlightening an investigation into less commonly used tools that weaving links do not evade. Recall that for  $p \geq 3$  odd and  $q \geq 2$ ,  $W(p, q)$  is an alternating strongly negative amphicheiral knot (SNAK), and is thus  $\mathbb{Q}$ -slice. Assume throughout this section that  $p \geq 3$  is odd and  $q \geq 2$ .

Ozsváth and Szabó’s correction term  $d$ , denoted  $d(Y, \mathfrak{t})$  [OS03a], in some way provides information about

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<sup>2</sup>Note that  $\det(W(7, 2)) = 169$ , and  $W(7, 2)$  is a known slice knot.

the intersection forms of 4-manifolds which bound a rational homology 3-sphere  $Y$  equipped with a  $\text{Spin}^c$  structure  $\mathfrak{t}$ . Manolescu and Owens [MO07] proved that it may be used to construct a concordance invariant of knots they call  $\delta(K)$ . By taking  $Y$  to be the double branched cover  $\Sigma(K)$  of  $S^3$  over a knot  $K$ , equipped with a  $\text{Spin}^c$  structure  $\mathfrak{t}_0$  induce by the unique  $\text{Spin}$  structure on  $\Sigma(K)$ , Manolescu and Owens define the invariant  $\delta(K)$  by

$$\delta(K) = 2d(\Sigma(K), \mathfrak{t}_0).$$

However, since our weaving links are rationally slice for  $p \geq 3$  odd and  $q \geq 2$ , then  $\Sigma(K)$  bounds a rational homology 4-ball  $W$ . Ozsváth and Szabó demonstrated this particular scenario:

**Proposition 3.9** (Ozsváth-Szabó, [OS03a] Prop. 9.9). If  $(Y, \mathfrak{t})$  is a rational homology three-sphere which bounds a rational homology four-ball  $W$ , so that  $\mathfrak{t}$  can be extended over  $W$ , then  $d(Y, \mathfrak{t}) = 0$ .

It follows that  $d(\Sigma(K), \mathfrak{t}_0) = 0$ , so the  $\delta$ -invariant vanishes for SNAKs. In fact, Manolescu and Owens [MO07] proved that if  $K$  is an alternating knot, then

$$\delta(K) = \sigma'(K),$$

where  $\sigma'(K) = -\frac{\sigma(K)}{2}$ . Hence, for all alternating SNAKs, such as weaving links with odd  $p$ ,  $\delta$  vanishes.

In 2014, Hom and Wu [HW16] developed the  $\nu^+$  invariant.  $\nu^+$  is a concordance invariant of knots obtained by examining the module structure of a certain  $\mathbb{Z} \oplus \mathbb{Z}$ -filtered chain complex over the ring  $\mathbb{F}_2[U, U^{-1}]$  developed using knot Floer homology. For more details, see [HW16]. More importantly,  $\nu^+$  provides us a better lower bound on the 4-genus of a knot. It also provides an upper bound on Ozsváth and Szabó's  $\tau$  invariant [OS03b], an integer-valued knot invariant that is computed to be the minimum integer  $m$  of an induced sequence of maps

$$\iota_K^m : H_*(\mathcal{F}(K, m)) \rightarrow \mathbb{Z}$$

such that  $\iota_K^m$  is nontrivial. For a proper view of the construction, see [OS03b].

**Proposition 3.10** (Hom-Wu, [HW16] Prop. 2.3). The invariant  $\nu^+$  is a concordance invariant satisfying the following inequality:

$$\nu^+(K) \geq \nu(K) \geq \tau(K).$$

Thus we have the inequality

$$\tau(K) \leq \nu^+(K) \leq g_4(K).$$

Hom and Wu demonstrate that  $\nu^+$  is determined completely by the knot signature in the case of alternating (and more generally, quasi-alternating) knots. In particular:

**Theorem 3.11** (Hom-Wu, [HW16] Thm. 2). Let  $K \subset S^3$  be a quasi-alternating knot. Then,

$$\nu^+(K) = \begin{cases} 0 & \text{if } \sigma(K) \geq 0, \\ -\frac{\sigma(K)}{2} & \text{if } \sigma(K) < 0. \end{cases}$$

As we demonstrated earlier, signature vanishes for SNAKs, such as weaving links with odd  $p$ . Thus, the  $\nu^+$  invariant (and consequently the  $\tau$  invariant) vanishes for alternating SNAKs, particularly for  $W(p, q)$ .

In 2019, Truong [Tru19] developed the  $\nu_n$  invariant, a concordance invariant whose computation was inspired by that of Ozsváth-Szabo and Hom-Wu before. By Propositions 3.7 and 3.8 of [Tru19], we obtain the following inequality:

$$\tau(K) \leq \nu(K) = \nu_1(K) \leq \dots \leq \nu_n(K) \leq \nu^+(K) \leq g_4(K).$$

By our results above,  $\nu_n(K)$  vanishes for all integers  $n$  when  $K$  is a SNAK.

In 2004, Rasmussen [Ras10] introduced the  $s$  invariant, which is constructed using spectral theory along with Khovanov homology. The  $s$  invariant forms a lower bound on slice genus. However, Rasmussen proved the following theorem:

**Theorem 3.12** (Rasmussen, [Ras10] Thm. 3). If  $K$  is an alternating knot, then  $s(K)$  is equal to the classical knot signature  $\sigma(K)$ .

As  $W(p, q)$  is both alternating and a SNAK,  $s(K) = \sigma(K) = 0$ .

Finally, we observe Hom's  $\epsilon$  invariant [Hom14].  $\epsilon(K)$  is a concordance invariant associated to the knot Floer complex of  $K$ . Originally developed for observing the behavior of the  $\tau$  invariant under knot cabling,  $\epsilon(K)$  is a strictly stronger invariant than  $\tau$  at obstructing sliceness. In particular, if  $K$  is slice, then  $\epsilon(K) = 0$ . There are two interesting ways that this invariant vanishes. Recall that if both  $p$  and  $q$  are odd, then  $W(p, q)$  is strongly *positively* amphicheiral. By Proposition 3.6 of [Hom14],  $\epsilon$  holds a mirroring property, given by

$$\epsilon(\overline{K}) = -\epsilon(K).$$

Now, if  $K$  is strongly positively amphicheiral, then

$$\epsilon(\overline{K}) = \epsilon(K).$$

Thus, we have

$$\epsilon(\overline{K}) + \epsilon(K) = 0 \implies \epsilon(K) + \epsilon(K) = 0 \implies \epsilon(K) = 0$$

Even if  $q$  is not odd, this invariant vanishes. Being an alternating knot,  $W(p, q)$  is *homologically thin*, as discovered by Manolescu and Ozsváth [MO08]. By Hom's Proposition 3.6 of [Hom14], if  $K$  is homologically thin, then

$$\epsilon(K) = \text{sgn } \tau(K)$$

We have shown that  $\tau$  vanishes for alternating SNAKs, so  $\epsilon$  vanishes as well.

While not an exhaustive list, we conclude that many of our stronger modern invariants appear to be unable to obstruct sliceness in strongly negative amphicheiral, alternating knots, such as  $W(p, q)$ . Any further investigation of sliceness obstructions for these knots would therefore require tools unperturbed by these properties of  $W(p, q)$ .

## 4 PROOFS OF THEOREMS

We now provide a proof for the formulas of the Seifert matrix and the Alexander polynomial in Theorems 2.3 and 3.5, and by extension Corollary 3.6.

### 4.1 Seifert Matrix Computation

We provide a proof for Theorem 2.3 by the application of the following lemma:

**Lemma 4.1.** Let  $p \in \mathbb{Z}$  with  $p \geq 3$ . Then the  $(p-1) \times (p-1)$  matrix  $A_{p,q}$  defined by

$$(A)_{ij} = \begin{cases} (-1)^{i+1}, & i = j \\ -1, & j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

is a Seifert matrix for  $W(p, 2)$ .

*Proof.* We use the principle of mathematical induction on the base case  $p = 3$ . Taking  $\Sigma_{3,2}$  to be our Seifert

surface for  $W(3, 2)$  as defined previously, we have  $H_1(\Sigma_{3,2}; \mathbb{Z})$  generators  $e_1$  and  $e_2$ . By direct computation of the Seifert form (See Figure 4.1), we have the matrix

$$A_{3,2} = \begin{bmatrix} \text{lk}(e_1, e_1^+) & (e_1, e_2^+) \\ \text{lk}(e_2, e_1^+) & \text{lk}(e_2, e_2^+) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}.$$

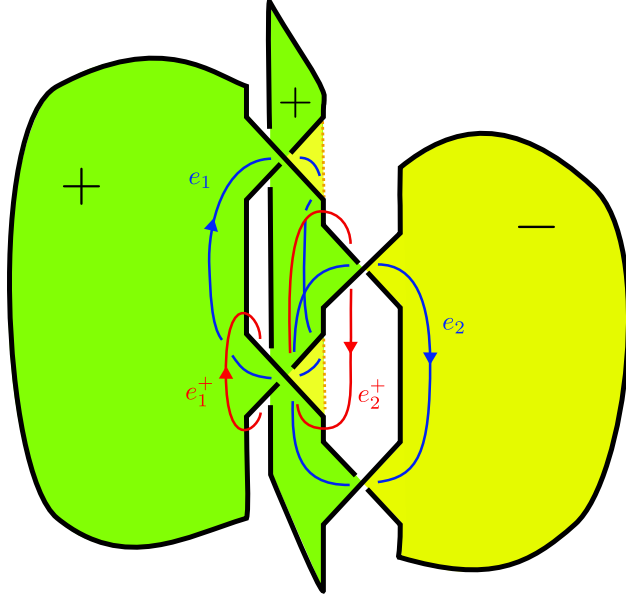


Figure 4.1: Seifert form computations for  $W(3, 2)$

Assume that for  $n = k$ ,  $A_{k,2}$  is a  $(k-1) \times (k-1)$  Seifert matrix for  $W(k, 2)$ , where

$$A_{k,2} = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & -1 & -1 & \ddots & \vdots \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & 0 & (-1)^k \end{bmatrix}$$

Now, Suppose we construct our Seifert matrix using the basis  $\{e_1, e_2, \dots, e_{k-1}\}$  for  $H_1(\Sigma_{k,2})$  as before. When attaching another strand to make this into a  $(k+1, 2)$ -weaving link, We are adding another Seifert circle to the bottom of the stack of existing circles along with two new half-twists attaching the  $(k+1)^{\text{th}}$  Seifert

circle to the  $k^{\text{th}}$  circle. Then we must add an additional cycle  $e_k$  to the basis of  $H_1(\Sigma_{k+1,2}; \mathbb{Z})$ . Then

$$\text{lk}(e_k, e_i^+) = \begin{cases} (-1)^{k+1}, & i = k \\ 0 & \text{otherwise} \end{cases}$$

and

$$\text{lk}(e_i, e_k^+) = \begin{cases} -1, & i = k - 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus, we indeed have that

$$A_{k+1,2} = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & -1 & -1 & \ddots & \vdots \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & 0 & (-1)^{k+1} \end{bmatrix}$$

as desired. This completes our process of induction.  $\square$

We now prove Theorem 2.3.

*Proof.* We use the principle of mathematical induction to form the Seifert matrix  $V_{p,q}$  on the base case  $q = 3$ . The structure of this base case may be easily generalized for our induction step. Let  $A_{p,2}$  be the Seifert matrix for  $W(p,2)$  as defined in Lemma 4.1, with homology basis  $\{e_1, e_2, \dots, e_{p-1}\}$ . Then the top left block of  $V_{p,3}$  is  $A_{p,2}$ . Take a copy of the braid  $(\sigma_1 \sigma_2^{-1} \dots \sigma_{p-1}^{\pm 1})$  and add it to the end of the braid for  $W(p,2)$  to form the braid word for  $W(p,3)$ . With the construction of our Seifert surface as before, we add to each Seifert circle one additional cycle to the basis for  $H_1(\Sigma_{p,3})$ , for a total of  $p - 1$  more generating cycles. Now, we see that the Seifert form  $\text{lk}(e_i, e_j^+) = \text{lk}(e_{i+p-1}, e_{j+p-1}^+)$  with  $e_i, e_j \in \{e_1, \dots, e_{p-1}\}$  so that we have the lower right block of  $V_{p,3}$  is  $A_{p,2}$ . We need only check the interactions between the newly added generating cycles  $\{e_p, e_{p+1}, \dots, e_{2p-1}\}$  from our braid word “extension” and the generating cycles  $\{e_1, e_2, \dots, e_{p-1}\}$  from our original braid word for  $W(p,2)$ . First, we can check the linking number of lengthwise cycles. For each  $i$ , we see the computations of  $\text{lk}(e_i, e_{i+p-1}^+)$  and  $\text{lk}(e_{i+p-1}, e_i^+)$  in Figure 4.2.

Then we have that

$$\text{lk}(e_i, e_{i+p-1}^+) = \begin{cases} 1, & 2|i \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \text{lk}(e_{i+p-1}, e_i^+) = \begin{cases} 0, & 2|i \\ -1, & \text{otherwise} \end{cases}$$

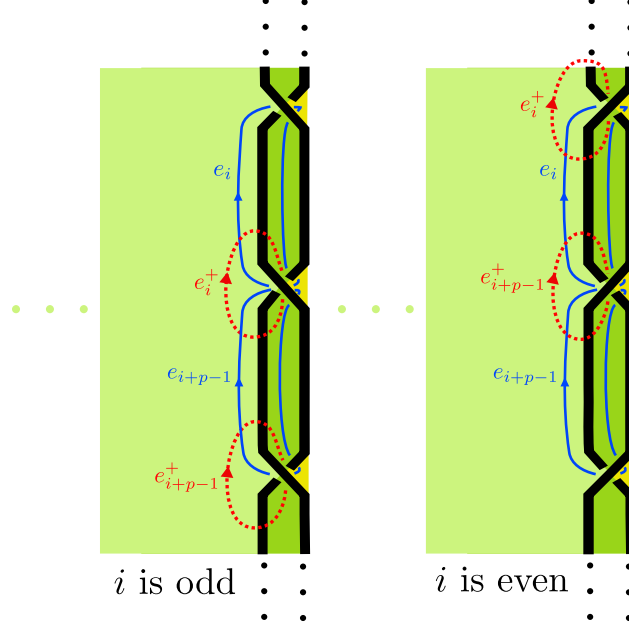


Figure 4.2: Linking numbers of lengthwise generators and their positive pushoffs

The only other nonzero linking numbers between lengthwise generators occurs from  $\text{lk}(e_{i+p}, e_i^+) = 1$ .

Therefore, the top right block of  $V_{p,3}$  is a  $p-1 \times p-1$  matrix  $B$  that satisfies

$$(B)_{ij} = \begin{cases} 1, & 2 \mid i \text{ and } i = j \\ 0, & \text{otherwise} \end{cases}$$

Similarly, the bottom left block will be a  $p-1 \times p-1$  matrix  $C$  satisfying

$$(C)_{ij} = \begin{cases} -1, & 2 \nmid i \text{ and } i = j \\ 1, & j = i + 1 \\ 0, & \text{otherwise} \end{cases}.$$

This forms our Seifert matrix  $V_{p,3}$  for  $W(p, 3)$ . This concludes our base case.

On the induction hypothesis that this holds for  $q = k$ , we can construct  $V_{p,k+1}$  for  $q = k + 1$  by a very similar argument by adding a new set of homology generators  $e_{kp}, e_{kp+1}, \dots, e_{(k+1)p-1}$  to the set  $\{e_1, e_2, \dots, e_{p-1}, \dots, e_{(k-2)p}, \dots, e_{(k-1)p-1}\}$  like before. For any two cycles  $e_i$  and  $e_j$  with  $|i - j| \geq q$ , there will be no nonzero interactions of any kind, so we insert a  $p-1$  zero matrix into that block of  $V_{p,k+1}$ .

As a result, we have for any  $p \geq 3$  and  $q \geq 2$  the block Seifert matrix  $V_{p,q}$  for  $W(p, q)$  defined by

$$(V_{p,q})_{i,j} = \begin{cases} A, & i = j \\ B, & j = i + 1 \\ C, & i = j + 1 \\ 0, & \text{otherwise} \end{cases}.$$

□

## 4.2 Alexander Polynomial

Using Theorem 2.3, we may construct explicitly the presentation matrix for  $W(p, q)$ :

**Corollary 4.2.** Let  $V_{p,q}$  be the block Seifert matrix as defined in Theorem 2.3. Then

$$V - tV^T = \begin{bmatrix} A - tA^T & B - tC^T & 0 & \dots & 0 \\ C - tB^T & A - tA^T & B - tC^T & \ddots & \vdots \\ 0 & C - tB^T & A - tA^T & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & B - tC^T \\ 0 & \dots & 0 & C - tB^T & A - tA^T \end{bmatrix}$$

is a presentation matrix for the Alexander module of  $W(p, q)$ , where

$$(A - tA^T)_{ij} = \begin{cases} (-1)^{i+1}(1-t) & i = j \\ -1 & j = i + 1 \\ t & i = j + 1 \\ 0 & \text{otherwise} \end{cases}, \quad (B - tC^T)_{ij} = \begin{cases} 1 & i = j, i \text{ is even} \\ t & i = j, i \text{ is odd} \\ -t & i = j + 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$(C - tB^T)_{ij} = \begin{cases} -t & i = j, i \text{ is even} \\ -1 & i = j, i \text{ is odd} \\ 1 & i = j + 1 \\ 0 & \text{otherwise} \end{cases}$$

We now present a proof for Theorem 3.5 and Corollary 3.6:

*Proof.* Let  $W(p, q)$  be a weaving link with  $p \geq 3$  and  $q \geq 2$ , and let  $S = V - tV^T$  be the presentation matrix for the Alexander module as defined in Corollary 4.2. Let  $L = A - tA^T$ ,  $M = B - tC^T$  and  $N = C - tB^T$ . Then  $S$  has the form of a block tridiagonal matrix. We note that  $M$  is an invertible matrix, being triangular with nonzero determinant. Molinari [Mol08] presented the following formula for the determinant of a block tridiagonal matrix:

$$\det(S) = (-1)^{(p-1)(q-1)} \det[T_{11}] \det[M^{q-2}]$$

where  $T_{11}$  is the  $(1, 1)$  entry of the block matrix

$$T = \begin{bmatrix} -L & -N \\ I & 0 \end{bmatrix} \underbrace{\begin{bmatrix} -M^{-1}L & -M^{-1}N \\ I & 0 \end{bmatrix} \cdots \begin{bmatrix} -M^{-1}L & -M^{-1}N \\ I & 0 \end{bmatrix}}_{q-2 \text{ times}} \begin{bmatrix} -M^{-1}L & -M^{-1}N \\ I & 0 \end{bmatrix}.$$

We construct a sequence to compute the  $(1, 1)$  entry. Let

$$X_0 = I, X_1 = -M^{-1}L, \text{ and } X_{k+1} = (-M^{-1}L)X_k - (M^{-1}N)X_{k-1}.$$

Note that  $L = -M - N$ . Therefore we have

$$X_0 = I, X_1 = I + M^{-1}N, \text{ and } X_{k+1} = (I + M^{-1}N)X_k - (M^{-1}N)X_{k-1}.$$

We show that  $X_n = \sum_{i=0}^n (M^{-1}N)^i$  by induction. First observe that this holds for our initial conditions:

$$X_0 = \sum_{i=0}^0 (M^{-1}N)^i = I \quad \text{and} \quad \sum_{i=0}^1 (M^{-1}N)^i = I + M^{-1}N$$

Assume that  $X_k = \sum_{i=0}^k (M^{-1}N)^i$  and  $X_{k-1} = \sum_{i=0}^{k-1} (M^{-1}N)^i$ . We have

$$\begin{aligned} X_{k+1} &= (I + M^{-1}N) \sum_{i=0}^k (M^{-1}N)^i - (M^{-1}N) \sum_{i=0}^{k-1} (M^{-1}N)^i \\ &= \sum_{i=0}^k (M^{-1}N)^i + \sum_{i=1}^{k+1} (M^{-1}N)^i - \sum_{i=1}^k (M^{-1}N)^i \\ &= \sum_{i=0}^k (M^{-1}N)^i + (M^{-1}N)^{k+1} \\ &= \sum_{i=0}^{k+1} (M^{-1}N)^i. \end{aligned}$$

Therefore,  $T_{11} = -LX_{q-2} - NX_{q-3} = -L \sum_{i=0}^{q-2} (M^{-1}N)^i - N \sum_{i=0}^{q-3} (M^{-1}N)^i$ . Further, we have

$$\begin{aligned}
T_{11} &= -L \sum_{i=0}^{q-2} (M^{-1}N)^i - N \sum_{i=0}^{q-3} (M^{-1}N)^i = (M + N) \sum_{i=0}^{q-2} (M^{-1}N)^i - N \sum_{i=0}^{q-3} (M^{-1}N)^i \\
&= M \sum_{i=0}^{q-2} (M^{-1}N)^i + N \sum_{i=0}^{q-2} (M^{-1}N)^i - N \sum_{i=0}^{q-3} (M^{-1}N)^i \\
&= M \left( \sum_{i=0}^{q-2} (M^{-1}N)^i - \sum_{i=1}^{q-2} (M^{-1}N)^i \right) + N \sum_{i=0}^{q-2} (M^{-1}N)^i \\
&= M + N \sum_{i=0}^{q-2} (M^{-1}N)^i \\
&= M \left( I + \sum_{i=1}^{q-1} (M^{-1}N)^i \right) \\
&= M \sum_{i=0}^{q-1} (M^{-1}N)^i.
\end{aligned}$$

As  $M$  is a triangular matrix, one observes that  $\det(M) = t^{\lfloor \frac{p-1}{2} \rfloor}$ . Therefore

$$\det(S) = (-1)^{(p-1)(q-1)} t^{\lfloor \frac{p-1}{2} \rfloor (q-1)} \det \left( \sum_{i=0}^{q-1} (M^{-1}N)^i \right).$$

Now,  $\Delta_{W(p,q)}(t) \doteq \det(S)$ , so we have

$$\Delta_{W(p,q)}(t) \doteq \det \left( \sum_{i=0}^{q-1} (M^{-1}N)^i \right)$$

as desired.

To satisfy Corollary 3.6, we need only show that  $I - M^{-1}N$  is invertible. That is, we show that the Laurent polynomial  $f(t) = \det(I - M^{-1}N)$  is not the zero polynomial. We simply evaluate the determinant of  $(I - M^{-1}N)(t)$  at  $t = 1$  and show that it is nonzero. Letting  $t = 1$ , we have

$$(I - M^{-1}N)(1) = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ 1 & 1 & -1 & \ddots & \vdots \\ 1 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 1 & 0 & \dots & 0 & 1 \end{pmatrix}$$

This determinant evaluates nicely as  $f(1) = p$  through a recursive operation. Therefore  $f(t)$  is not the

zero polynomial, and  $I - M^{-1}N$  is invertible. Therefore, we have

$$\Delta_{W(p,q)}(t) \doteq \det \left( \sum_{i=0}^{q-1} (M^{-1}N)^i \right) = \frac{\det(I - (M^{-1}N)^q)}{\det(I - M^{-1}N)}$$

as desired. □

## 5 CONCLUSION

We sought to find reliable obstructions to sliceness in the class of weaving knots. We have shown that if a weaving knot  $W(p, q)$  with  $\gcd(p, q) = 1$  satisfies either

- $p$  is an even integer
- $q = 2$  and  $p \neq 7$

then  $W(p, q)$  cannot be slice. We proved these using knot signature and knot determinant. Examples of non-slice weaving knots that do not fit these categories exist (Consider  $W(3, 5)$  and  $W(5, 3)$  in Figure 2.6 as examples). However, these examples are sporadic and do not carry implications for further examples of non-slice weaving knots. Since we restrict our view to all weaving knots with  $p \geq 3$  odd, the remaining weaving knots to check are all alternating SNAKs. We demonstrated how many of our previously reliable obstructions to sliceness vanish due to these two properties of symmetry. It is striking how such a simple knot construction can disrupt modern approaches to obstructing sliceness in this way.

### 5.1 Open Questions

There remains many ways in which we may expand upon our results. Let  $p \geq 3$  and  $q \geq 2$ .

Our attempts at computing the Alexander polynomial are in a way beneficial, though it would be convenient to have a closed formula.

**Problem 5.1.** Find a closed form for  $\Delta_{W(p,q)}(t)$  (as in Corollary 3.6) in terms of  $p$  and  $q$ .

The problem lies in the denseness of  $(M^{-1}N)^q$ . Once Conjecture 3.7 is proven, one would only need to simplify the determinant computation of  $I - (M^{-1}N)^q$ . Perhaps an investigation into Burau representations [Bur35] could lend some results.

We once again direct readers to the efforts and results of Dowdall et. al [DMMS10] in computing the determinant of weaving links. While the only immediate results showed implications for  $W(p, 2)$ , perhaps a computation of the remaining weaving link determinants could lend results:

**Problem 5.2.** Let  $\gcd(p, q) = 1$ . Compute  $\det(W(p, q))$  for the case that  $p \geq 5$  is odd and  $q \geq 3$ .

As discussed, modern invariants that are sensitive to symmetry are likely to fail at obstructing sliceness in weaving knots. We must then determine invariants that are unperturbed by these properties.

**Problem 5.3.** Compute Casson-Gordon invariants [CG78] for  $W(p, q)$ .

Certain Casson-Gordon invariants of alternating SNAKs, specifically  $\sigma(K, \chi)$  [CG86], may vanish like before. However, we have not explored these invariants thus far.

Another potential direction may lie in some equivariant Floer theory. Let  $K$  be a knot with exterior  $E(K)$ . If we equip  $E(K)$  with an involution in the form of a  $\mathbb{Z}_2$  group action, we can in some way investigate how many properties of the space are affected. This is known as *involutive Floer theory*, and we direct readers to the first work of Hendericks, Lipshitz, and Sarkar [HLS16]. If instead we equip  $E(K)$  with some other Lie group, particularly a dihedral group  $D_{2n}$ , it is possible that there may exist invariants that encode the symmetric properties such as amphicheirality or invertibility into their information. While some results may be found here, not much research has been done on equivariant Floer theory for groups other than  $\mathbb{Z}_2$ . Perhaps new invariants in this field may be computed that are unperturbed by amphicheirality or invertibility as the ones we have explored. This would require much additional study and time without a proper background.

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# APPENDIX

## DETERMINATION OF RESEARCH (DOR) APPROVAL



Parker Smiley <pds69860@ucmo.edu>

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### Determination of Research - IRB Protocol Not Required

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Kathy Schnakenberg via Smartsheet <automation@app.smartsheet.com>  
Reply-To: Kathy Schnakenberg <schnakenberg@ucmo.edu>  
To: pds69860 <pds69860@ucmo.edu>

Fri, Sep 12, 2025 at 9:29 AM



### Determination of Research - IRB Protocol Not Required

Dear Investigator,

Your project (Exploring Open Questions on Weaving Links and Knots) does not require review by IRB. You may proceed with your project. If you need to provide verification of your submission to IRB, please download or print a copy of this email response.

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