9.4 The Cross Product

In this section, we will examine the cross product

Definition and Properties

Algebraically, the cross product of two vectors $\mathbf{v} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ and $\mathbf{w} = d\hat{\mathbf{i}} + e\hat{\mathbf{j}} + f\hat{\mathbf{k}}$, is

$$\mathbf{v} \times \mathbf{w} := (bf - ce)\hat{\mathbf{i}} - (af - cd)\hat{\mathbf{j}} + (ae - bd)\hat{\mathbf{k}}.$$

Geometrically, the length of $\mathbf{v} \times \mathbf{w}$ is $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin(\theta)$, where θ is the angle between the two vectors $\mathbf{v}$ and $\mathbf{w}$. Further, this length corresponds to the area of the parallelogram formed from the two vectors $\mathbf{v}$ and $\mathbf{w}$, as in figure shown in the textbook.

Question 1. Compute the cross product of $3\hat{\mathbf{i}}+5\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$. What is the area of the parallelogram created from these two vectors?

Question 2. Let $A = (2, 3, 0)$, $B = (-1, 1, 1)$, and $C = (0, 1, -2)$. Consider the triangle ABC

(a) Find the angle the triangle ABC has at vertex A.

(b) Find the area of the triangle ABC.

Question 3. Suppose that $\mathbf{v} \cdot \mathbf{w} = 6$ and $\|\mathbf{v} \times \mathbf{w}\| = 3$. Find $\tan\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

Question 4. Note that if two vectors are parallel, then the angle between them is either $\theta = 0$ or $\theta = \pi$.

(a) What is $\sin(0)$? $\sin(\pi)$?

(b) If $\mathbf{v}$ and $\mathbf{w}$ are parallel, what can you say about $\|\mathbf{v} \times \mathbf{w}\|$? What about $\mathbf{v} \times \mathbf{w}$?

Parallel Property
If $\mathbf{v}$ and $\mathbf{w}$ are parallel, then $\mathbf{v} \times \mathbf{w} = \mathbf{0}$.

Question 5. Compute $\mathbf{v} \times \mathbf{v}$.

Directional Property

Should two vectors $\mathbf{v}$ and $\mathbf{w}$ *not* be parallel, then the vector $\mathbf{v} \times \mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ with direction respecting the right-hand rule:

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \qquad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} \qquad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$
$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}} \qquad \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}} \qquad \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$$

Question 6. Compute the following using the right-hand rule:

(a) $(\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) \times \hat{\mathbf{k}}$

(b) $\hat{\mathbf{k}} \times -\mathbf{i}$

(c) $\frac{-1}{2}\hat{\mathbf{j}} \times 2\hat{\mathbf{k}}$

Question 7. Use the right hand rule to prove each of the following:

(a) $\left(\hat{\mathbf{i}} \times \hat{\mathbf{j}}\right) \times \hat{\mathbf{k}} = \hat{\mathbf{i}} \times \left(\hat{\mathbf{j}} \times \hat{\mathbf{k}}\right)$.

(b) Show that the cross product is not "associative" by showing that $\left(\hat{\mathbf{i}} \times \hat{\mathbf{j}}\right) \times \left(\hat{\mathbf{j}} \times \hat{\mathbf{k}}\right) \neq \hat{\mathbf{i}} \times \left(\left(\hat{\mathbf{j}} \times \hat{\mathbf{j}}\right) \times \hat{\mathbf{k}}\right)$. That is, show that the order in which we perform various cross produicts matter!

Question 8. Using all properties discussed in this section of the workbook, what can you say about the relationship between $\mathbf{v} \times \mathbf{w}$ and $\mathbf{w} \times \mathbf{v}$. As we are discussing vectors, your answer should address BOTH direction and magnitude.