9.2 Vectors

In this section, we will introduce the notion of a vector, and study their algebraic and geometric structures.

Vectors

A n**-dimensional vector** is a list $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of n real numbers. We interpret a vector as an arrow pointing from the origin $\mathbf{0} = (0, 0, \ldots, 0)$ to the point $p_{\mathbf{x}} = (x_1, x_2, \ldots, x_n)$.

The collection of all n-dimensional vectors is called a n**-dimensional real vector space**, or just n**-dimensional space** for short, often written $\mathbb{R}^n$.

Given two vectors $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ in $\mathbb{R}^n$, we can add them them together:

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, \ldots, v_n + w_n).$$

Geometrically, this corresponds to put the second vector's "tail" at the "tip" of the first vector.

Further, we can scale a vector by a **scalar**, λ (a real number):

$$\lambda\mathbf{v} = (\lambda v_1, \ldots, \lambda v_n).$$

Geometrically, scaling cooresponds to stretching or compressing the line segment $\overline{\mathcal{O}P_{\mathbf{v}}}$ by a factor of λ, with reflection if $\lambda < 0$.

The **length** or **norm** of a vector $\mathbf{v} = (v_1, \ldots, v_n)$ is the quantity $\|\mathbf{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}$, which is also often written as $|\mathbf{v}|$. We say a vector is a unit vector if $\|\mathbf{v}\| = 1$. The norm satisfies two important properties: the triangle inequality

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|,$$

and the scalar law:

$$\|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\|.$$

To get the **unit vector** pointing in the same direction as the non-zero vector $\mathbf{v}$, we can divide $\mathbf{v}$ by $\|\mathbf{v}\|$, i.e. $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$.

Question 1. In three dimensions, we often write vectors as a sum of the basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, where $\hat{\mathbf{i}}$ is the vector $(1, 0, 0)$, $\hat{\mathbf{j}}$ is the vector $(0, 1, 0)$, and $\hat{\mathbf{k}}$ is the vector $(0, 0, 1)$. For each of the following either convert the vector from list form to $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ form or vice versa. Then, compute the norm of the vector.

(a) $(0, -1, 2)$

(b) $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$

(c) $\hat{\mathbf{i}} - \hat{\mathbf{k}}$

(d) $(3, 2, -5)$

Question 2. Find all of the 2-dimensional vectors having norm 13 with y-component 5. (Hint: Use the definition of the norm in $\mathbb{R}^2$.)

Question 3. Find all of the 2-dimensional vectors having norm k (here $k > 0$) that lie on the line $x+y=0$.