9.2 Vectors

In this section, we will introduce the notion of a vector, and study their algebraic and geometric structures.

Vectors

A n**-dimensional vector** is a list $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of n real numbers. We interpret a vector as an arrow pointing from the origin $\mathbf{0} = (0, 0, \ldots, 0)$ to the point $p_{\mathbf{x}} = (x_1, x_2, \ldots, x_n)$.

The collection of all n-dimensional vectors is called a n**-dimensional real vector space**, or just n**-dimensional space** for short, often written $\mathbb{R}^n$.

Given two vectors $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ in $\mathbb{R}^n$, we can add them them together:

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, \ldots, v_n + w_n).$$

Geometrically, this corresponds to put the second vector's "tail" at the "tip" of the first vector.

Further, we can scale a vector by a **scalar**, λ (a real number):

$$\lambda\mathbf{v} = (\lambda v_1, \ldots, \lambda v_n).$$

Geometrically, scaling cooresponds to stretching or compressing the line segment $\overline{OP_{\mathbf{v}}}$ by a factor of λ, with reflection if $\lambda < 0$.

The **length** or **norm** of a vector $\mathbf{v} = (v_1, \ldots, v_n)$ is the quantity $\|\mathbf{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}$, which is also often written as $|\mathbf{v}|$. We say a vector is a unit vector if $\|\mathbf{v}\| = 1$. The norm satisfies two important properties: the triangle inequality

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|,$$

and the scalar law:

$$\|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\|.$$

To get the **unit vector** pointing in the same direction as the non-zero vector $\mathbf{v}$, we can divide $\mathbf{v}$ by $\|\mathbf{v}\|$, i.e. $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$.

Question 1. In three dimensions, we often write vectors as a sum of the basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, where $\hat{\mathbf{i}}$ is the vector $(1, 0, 0)$, $\hat{\mathbf{j}}$ is the vector $(0, 1, 0)$, and $\hat{\mathbf{k}}$ is the vector $(0, 0, 1)$. For each of the following either convert the vector from list form to $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ form or vice versa. Then, compute the norm of the vector.

(a) $(0, -1, 2)$

(b) $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$

(c) $\hat{\mathbf{i}} - \hat{\mathbf{k}}$

(d) $(3, 2, -5)$

Question 2. Find all of the 2-dimensional vectors having norm 13 with y-component 5. (Hint: Use the definition of the norm in $\mathbb{R}^2$.)

Question 3. Find all of the 2-dimensional vectors having norm k (here $k > 0$) that lie on the line $x+y=0$.

9.3 The Dot Product and Projections

In this section, we look at the definition and uses of the dot product

Definition and Properties

Algebraically, the dot product of two vectors, $\mathbf{v} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ and $\mathbf{w} = d\hat{\mathbf{i}} + e\hat{\mathbf{j}} + f\hat{\mathbf{k}}$, is

$$\mathbf{v} \cdot \mathbf{w} := ad + be + cf.$$

Geometrically, the dot product of $\mathbf{v}$ and $\mathbf{w}$ is the product of their lengths and the cosine of the angle between them:

$$\mathbf{v} \cdot \mathbf{w} := \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$$

where θ is the angle between the two vectors.

The most important property of the dot product is:

$$\mathbf{v} \cdot \mathbf{w} = 0 \text{ if and only if } \mathbf{v} \text{ is perpendicular to } \mathbf{w}.$$

Question 1. Compute the dot product of $3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$. What is the angle between these vectors?

Question 2. What $\mathbf{v} \cdot \mathbf{v}$? (What is the angle of between a vector and itself?)

Question 3. Is the vector from $(4, 1, 2)$ to $(2, 4, 3)$ perpendicular to the vector $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$?

Question 4. The vectors $3\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} - y\hat{\mathbf{j}}$ are perpendicular. Use the dot product to find y.

Question 5. Given vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ so that $\mathbf{u} \cdot \mathbf{v} = 2$, $\mathbf{u} \cdot \mathbf{w} = -3$, $\|\mathbf{u}\| = 5$, $\|\mathbf{v}\| = 1/4$, and $\|\mathbf{w}\| = 1$, compute the following:

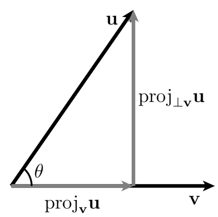
(a) $2\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$

(b) The angle between $\mathbf{v}$ and $\mathbf{w}$.

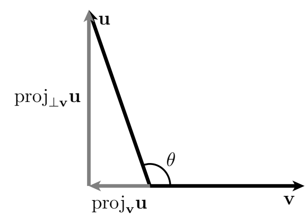
(c) If $(2\mathbf{u} + \mathbf{v}) \cdot (2\mathbf{w}) = -6$, find $\mathbf{v} \cdot \mathbf{w}$.

Projection of one vector onto another

Given a vector $\mathbf{v}$, we want to know 'how much' of another vector $\mathbf{u}$ points in the direction of $\mathbf{v}$. This is called the projection of $\mathbf{u}$ onto $\mathbf{v}$ and it is a multiple of $\mathbf{u}$, as you can see in the following diagram:



Left: $\text{proj}_{\mathbf{v}}\,\mathbf{u}$,



Right: $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ when $\theta > \frac{\pi}{2}$

Algebraically,

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{v}\|}\hat{\mathbf{v}}$$

Question 6. Compute $\text{proj}_{\mathbf{v}}(\mathbf{u})$ for the following choices of $\mathbf{u}$ and $\mathbf{v}$:

(a) $\mathbf{u} = 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{v} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$.

(b) $\mathbf{u} = 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{v} = -2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}d - 4\hat{\mathbf{k}}$.

(c) $\mathbf{u} = 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{v} = \hat{\mathbf{i}}$.

(d) $\mathbf{u} = 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{v} = 5\hat{\mathbf{i}}$.

9.4 The Cross Product

In this section, we will examine the cross product

Definition and Properties

Algebraically, the cross product of two vectors $\mathbf{v} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ and $\mathbf{w} = d\hat{\mathbf{i}} + e\hat{\mathbf{j}} + f\hat{\mathbf{k}}$, is

$$\mathbf{v} \times \mathbf{w} := (bf - ce)\hat{\mathbf{i}} - (af - cd)\hat{\mathbf{j}} + (ae - bd)\hat{\mathbf{k}}.$$

Geometrically, the length of $\mathbf{v} \times \mathbf{w}$ is $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin(\theta)$, where θ is the angle between the two vectors $\mathbf{v}$ and $\mathbf{w}$. Further, this length corresponds to the area of the parallelogram formed from the two vectors $\mathbf{v}$ and $\mathbf{w}$, as in figure shown in the textbook.

Question 1. Compute the cross product of $3\hat{\mathbf{i}}+5\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$. What is the area of the parallelogram created from these two vectors?

Question 2. Let $A = (2, 3, 0)$, $B = (-1, 1, 1)$, and $C = (0, 1, -2)$. Consider the triangle ABC

(a) Find the angle the triangle ABC has at vertex A.

(b) Find the area of the triangle ABC.

Question 3. Suppose that $\mathbf{v} \cdot \mathbf{w} = 6$ and $\|\mathbf{v} \times \mathbf{w}\| = 3$. Find $\tan\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

Question 4. Note that if two vectors are parallel, then the angle between them is either $\theta = 0$ or $\theta = \pi$.

(a) What is $\sin(0)$? $\sin(\pi)$?

(b) If $\mathbf{v}$ and $\mathbf{w}$ are parallel, what can you say about $\|\mathbf{v} \times \mathbf{w}\|$? What about $\mathbf{v} \times \mathbf{w}$?

Parallel Property
If $\mathbf{v}$ and $\mathbf{w}$ are parallel, then $\mathbf{v} \times \mathbf{w} = \mathbf{0}$.

Question 5. Compute $\mathbf{v} \times \mathbf{v}$.

Directional Property

Should two vectors $\mathbf{v}$ and $\mathbf{w}$ *not* be parallel, then the vector $\mathbf{v} \times \mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ with direction respecting the right-hand rule:

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \qquad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} \qquad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$
$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}} \qquad \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}} \qquad \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$$

Question 6. Compute the following using the right-hand rule:

(a) $(\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) \times \hat{\mathbf{k}}$

(b) $\hat{\mathbf{k}} \times -\mathbf{i}$

(c) $\frac{-1}{2}\hat{\mathbf{j}} \times 2\hat{\mathbf{k}}$

Question 7. Use the right hand rule to prove each of the following:

(a) $\left(\hat{\mathbf{i}} \times \hat{\mathbf{j}}\right) \times \hat{\mathbf{k}} = \hat{\mathbf{i}} \times \left(\hat{\mathbf{j}} \times \hat{\mathbf{k}}\right)$.

(b) Show that the cross product is not "associative" by showing that $\left(\hat{\mathbf{i}} \times \hat{\mathbf{j}}\right) \times \left(\hat{\mathbf{j}} \times \hat{\mathbf{k}}\right) \neq \hat{\mathbf{i}} \times \left(\left(\hat{\mathbf{j}} \times \hat{\mathbf{j}}\right) \times \hat{\mathbf{k}}\right)$. That is, show that the order in which we perform various cross produicts matter!

Question 8. Using all properties discussed in this section of the workbook, what can you say about the relationship between $\mathbf{v} \times \mathbf{w}$ and $\mathbf{w} \times \mathbf{v}$. As we are discussing vectors, your answer should address BOTH direction and magnitude.