11.8 Triple Integrals in Cylindrical and Spherical Coordinates

In this section, we will explore how to compute triple integrals in Cylindrical and Spherical coordinates.

Concept: Coordinate Systems on $\mathbb{R}^3$

To convert between Cartesian (x, y, z) coordinates and Cylindrical (r, θ, z) coordinates:

$$\begin{aligned} (r,\theta,z) &\Rightarrow (x,y,z) \\ x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned} \qquad \begin{aligned} (x,y,z) &\Rightarrow (r,\theta,z) \\ r^2 &= x^2+y^2 \\ \tan\theta &= \frac{y}{x} \\ z &= z \end{aligned}$$

To convert between Cartesian (x, y, z) coordinates and Spherical (ρ, θ, φ) coordinates:

$$\begin{aligned} (\rho,\theta,\varphi) &\Rightarrow (x,y,z) \\ x &= \rho\cos\theta\sin\varphi \\ y &= \rho\sin\theta\sin\varphi \\ z &= \rho\cos\varphi \end{aligned} \qquad \begin{aligned} (x,y,z) &\Rightarrow (\rho,\theta,\varphi) \\ \rho^2 &= x^2+y^2+z^2 \\ \tan\theta &= \frac{y}{x} \\ \tan\varphi &= \frac{\sqrt{x^2+y^2}}{z} \end{aligned}$$

To convert between Cylindrical (r, θ, z) coordinates and Spherical (ρ, θ, φ) coordinates:

$$\begin{aligned} (\rho,\theta,\varphi) &\Rightarrow (r,\theta,z) \\ r &= \rho\sin\varphi \\ \theta &= \theta \\ z &= \rho\cos\varphi \end{aligned} \qquad \begin{aligned} (r,\theta,z) &\Rightarrow (\rho,\theta,\varphi) \\ \rho^2 &= r^2+z^2 \\ \theta &= \theta \\ \tan\varphi &= \frac{r}{z} \end{aligned}$$

Further, the differential of volume can be expressed in any of these coordinate systems as

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2\sin(\varphi)\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z\,.$$

Question 1. **(Warmup:)** Find eqivalent coordinates in each of the three coordinate systems (Cartesian, Cylindrical, and Spherical) for each of the following points:

(a) $(x, y, z) = (1, 1, 1)$

(b) $(r, \theta, z) = (2, 3\pi/4, 1)$

(c) $(\rho, \theta, \varphi) = (11/7, 5\pi/12, -\pi/6)$

(d) $(x, y, z) = \left(-2, 3, -\sqrt{12}\right)$

Question 2. First, sketch the region in question. Then, write three triple integrals (one in each of our three coordinate systems) that yields the specified volume. Finally, choose one integral for each part and evaluate it.

(a) The portion of the ball $x^2 + y^2 + z^2 \leq 4$ above the plane $z = \sqrt{3}$.

(b) The cone $C = \left\{(r, \theta, z) : 0 \leq z \leq 16,\ 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq \frac{z}{4}\right\}$

(c) The cake slice $\left\{(r, \theta, z) : 0 \leq z \leq 4,\ 0 \leq r \leq 4,\ 0 \leq \theta \leq \frac{\pi}{6}\right\}$

Question 3. Let R_1 be the solid cone $\left\{(r,\theta,z) : -4 \leq z \leq 0,\ 0 \leq r \leq \frac{z+4}{2}\right\}$ and let R_2 be the solid hemisphere of radius 2 centered at the origin $R_2 = \left\{(\rho,\theta,\varphi) : 0 \leq \rho \leq 2,\ 0 \leq \varphi \leq \frac{\pi}{2}\right\}$. The region R is the union of R_1 and R_2. Find the volume of the region R.

Question 4. Evaluate each of the following integrals. You may want to change coordinate systems to make the integral easier!

(a) $\displaystyle\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xz \, \mathrm{d}z \, \mathrm{d}x \, \mathrm{d}y$

(b) $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}}\left(x^2+y^2+z^2\right)^2\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$

(c) $\displaystyle\iiint_B \frac{30}{\rho^2}\cos^2\theta\sin^4\varphi\,\mathrm{d}V$, where B is a sphere of radius 5 centered at the origin.

9.6 Vector-Valued Functions

9.6.1 Introduction to Vector-Valued Functions

In this section, we will begin exploring functions whose codomain have dimension larger than 1.

Main Concepts

A **vector-valued function** is a function whose input is a single real number t, and whose output is a a vector that depends on t. The **graph** of a vector-valued function is the set of all terminal points of the output vectors with their initial point at the origin. A **parametric curve** (or **parametric equation**) is a vector-valued function of the form

$$\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} \qquad \text{or} \qquad \mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$$

where the **component functions** $x(t)$, $y(t)$, and $z(t)$ are real-valued functions.
A **parametric line** is a vector-valued function $\mathbf{r}(t)$ of the form $\mathbf{r}(t) = \mathbf{b} + t\mathbf{v}$, where $\mathbf{b}$ and $\mathbf{v}$ are fixed vectors. We also say that this is a line **starting at b, in the direction of v**.

Question 1. In this problem, we will generalize the process of finding the equation of a 2-dimensional line to the process of finding the (parametric) equation of a line in higher dimensions.

(a) Let (x_0, y_0) and (x_1, y_1) be two points in $\mathbb{R}^2$, and assume that $x_0 \neq x_1$. Find a linear function that passes through these points. (**Hint:** point-point form)

(b) Now, (x_0, y_0) and (x_1, y_1) be two points in $\mathbb{R}^2$, but don't assume that $x_0 \neq x_1$. We want to find a parametric function $\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}$ such that $\mathbf{r}(0) = (x_0, y_0)$ and $\mathbf{r}(1) = (x_1, y_1)$.

Suppose $x(t) = a + bt$ and $y(t) = c + dt$. Then, we want $x(0) = x_0$ and $x(1) = x_1$ and similarly for y. Find values of a, b, c, d that satisfy the equations above.

(c) Using your function from the last part, solve $y = y(t)$ and $x = x(t)$ for t.

(d) Equate the two expressions you found in the last part to get a single equation involving only x and y. What do you notice about this equation?

(e) Do part (b) for the case where (x_0, y_0, z_0) and (x_1, y_1, z_1) are two points in $\mathbb{R}^3$.

Question 2. Find a parameterization of a circle of radius 11, contained in the plane $z = 0$ centered at the point $(1, -1, 0)$, and oriented clockwise.

Question 3. Find a vector-valued function $\mathbf{r}(t)$ that parameterizes the line through the point $(-2, 1, 4)$ in the direction of the vector $\mathbf{b} = \langle 3, 2, -5 \rangle$.

Question 4. Find a vector-valued function $\mathbf{r}(t)$ that parameterizes the line of intersection of the planes $x + 2y - z = 4$ and $3x + y - 2z = 1$.

Question 5.

(a) Determine the point of intersection of the lines given by $\mathbf{r}(t) = \langle 2, 1, 0\rangle + \langle 1, -2, 4\rangle t$ and $\mathbf{s}(t) = \langle 3, 3, 0\rangle + \langle 1, -2, 2\rangle t$.

(b) Then, find a vector valued function $\mathbf{q}(t)$ that parameterizesthe line that passes through the point of intersection and is perpendicular to the lines traced out by $\mathbf{r}(t)$ and $\mathbf{s}(t)$.