11.5 Double Integrals in Polar Coordinates

In this section, we will explore how to compute double integrals in polar (r, θ) coordinates.

Concept: Polar-Cartesian Coordinate Conversion

To convert from polar (r, θ) coordinates to Cartesian (x, y) coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

To convert form Cartesian (x, y) coordinates to polar (r, θ) coordinates:

$$r^2 = x^2 + y^2$$
$$\tan(\theta) = \frac{y}{x}$$

Further, the differential of area can be defined as $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$ or $\mathrm{d}A = r\,\mathrm{d}\theta\,\mathrm{d}r$

Question 1. **(Warm-Up:)** For parts (a)-(d), convert from polar coordinates to Cartesian coordinates. For parts (e)-(h), convert from Cartesian to polar coordinates.

(a) $(1, \pi/4)$

(b) $(2, 3\pi/4)$

(c) $(11/7, 5\pi/12)$

(d) $(5, 3)$

(e) $(5, 12)$

(f) $(3, 4)$

Question 2. For each of the below, convert the integrand to polar coodinates, then evaluate the double integral over the given polar region.

(a) $\iint_D x^2 + y^2 \, \mathrm{d}A$, where D is the region $3 \leq r \leq 5, 0 \leq \theta \leq 2\pi$.

(b) $\iint_D 1 + 2\arctan\left(\frac{y}{x}\right) \mathrm{d}A$, where D is the region $1 \leq r \leq 2$, $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$.

(c) $\iint_D x^4 + 2x^2y^2 + y^4 \, \mathrm{d}A$ where D is the region $3 \leq r \leq 4$, $\frac{-\pi}{6} \leq \theta \leq \frac{5\pi}{6}$

Question 3. In this question, we will "do the impossible" and integrate the Gaussian function $g(t) = e^{-t^2}$, which cannot be done using methods you learned in Calculus 2.

(a) First, we turn one integral into two by doing a clever trick. Observe that $\left(\int_{-\infty}^{\infty} e^{-t^2}\,\mathrm{d}t\right)^2$ can be rewritten as the product of two integrals:

$$\left(\int_{-\infty}^{\infty} e^{-t^2}\,\mathrm{d}t\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)\cdot\left(\int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y\right)$$

Write the right-hand side of this equation as a double integral in the variables x and y.

(b) Now, convert your new integral into polar coordinates. (Don't forget to change your bounds of integration!)

(c) For some choice of u, this integral can be transformed to be in the form $\int_0^{2\pi}\int_{-\infty}^{0} e^u\,\mathrm{d}u\,\mathrm{d}\theta$. Find such a u and verify this claim.

(d) Evaluate the integral $\int_0^{2\pi} \int_{-\infty}^{0} e^u \, \mathrm{d}u \, \mathrm{d}\theta$.

(e) Now, recalling that the value you computed is the value $\left(\int_{-\infty}^{\infty} e^{-t^2} \, \mathrm{d}t \right)^2$, compute the actual value of $\int_{-\infty}^{\infty} e^{-t^2} \, \mathrm{d}t$. (Hint: the value of the integral should be positive, as $e^{-t^2} > 0$ for all t).

Question 4. Consider the region $\Omega \subseteq \mathbb{R}^2$, where Ω is the region inside the curve $r = 1$ and to the right of the polar curve $r = \frac{1}{6}\sec\theta$.

(a) Sketch the region Ω.

(b) Set up and evaluate a double integral in polar coordinates to find the area of Ω.

(c) What is the average value of the function $f(x, y) = x^2 + y^2$ over the region Ω?