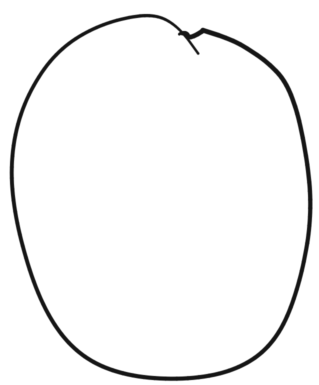
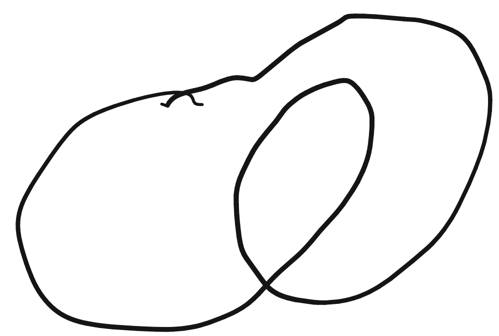
AC 12.8 / OS 6.4 || Green's Theorem

Def: A closed curve C is simple if it has no self-intersections

a simple closed curve
SCC



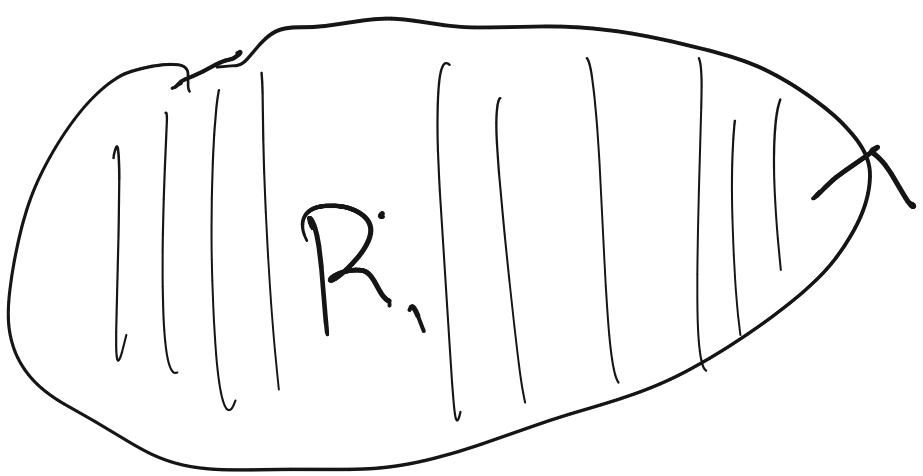
Simple



not
Simple

Theorem: Jordan Curve Theorem:

If C is a simple closed curve in $\mathbb{R}^2$ then C separates the plane into two regions, R_1, R_2 one of which is bounded.



R_2

If C is an oriented SCC, ...

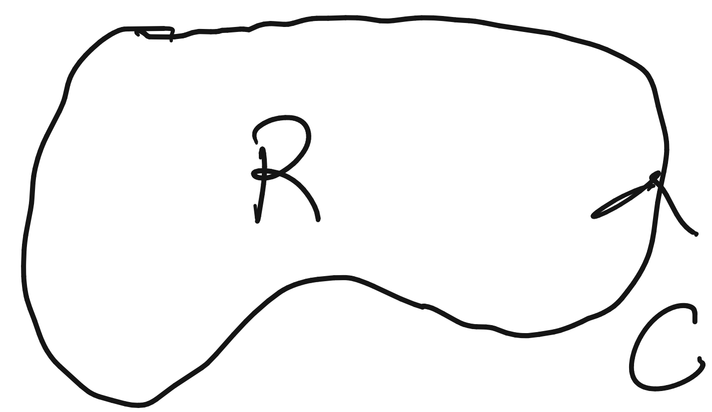
one of the regions "agrees" w/ C.

Def: let $\vec{F}$ a 2D v. field $F = \langle P, Q \rangle$

defined over a closed, Simply connected region R

in the plane.

Orient the bounding curve



C of R st R is "on the left" of C.

The Circulation density of F over R is

$\oint_C \vec{F} \cdot d\vec{r}$.

w/ same setup, the <u>Circulation</u> of F is

$$\iint_R (Q_x - P_y)\, dA = \iint_R \text{Curl}(F)\, dA$$

2D so Curl (F) can be treated as a <u>Scalar</u>!

Thm (<u>Green's theorem</u>) w/ the setup as above:

$F = \langle P, Q \rangle$ a 2D v. field

R a closed, simply conn. region

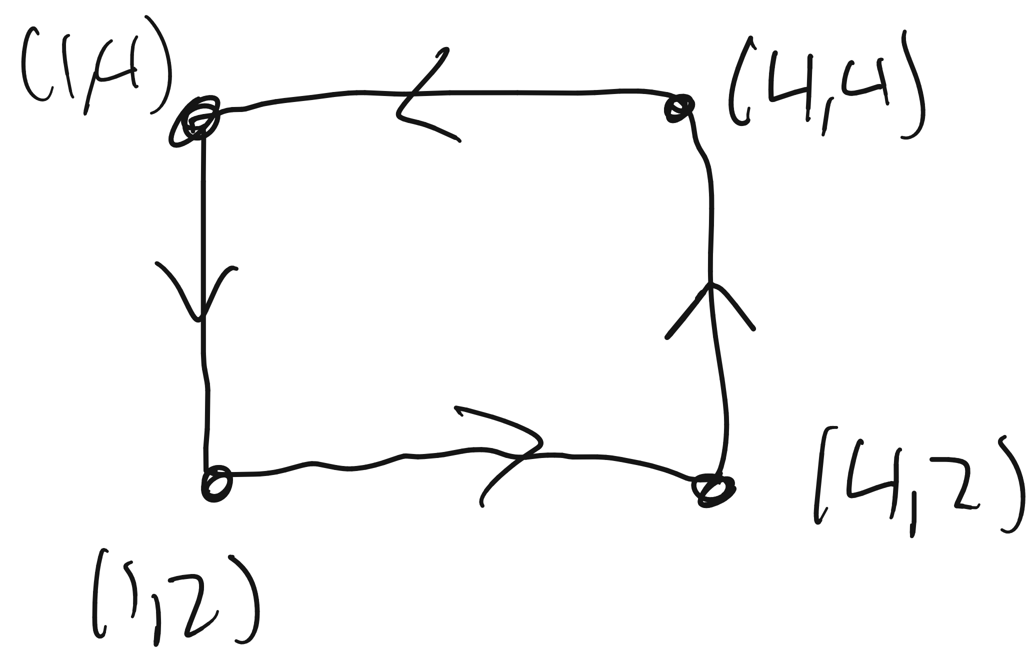
C = boundary of R oriented st R is "on the left"

We have

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_R (Q_x - P_y)\, dA = \iint_R \text{Curl}(\vec{F})\, dA}$$

<u>Ex ①</u> $F = \langle -3xy^5, 4y^9 \rangle$

$R =$



$$\begin{aligned} \text{Curl}(F) &= Q_x - P_y \\ &= 0 + 15xy^4 \\ &= 15xy^4 \end{aligned}$$

Green's Thm!

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}(F)\, dA$$

$$= \int_1^4 \int_2^4 15xy^4 \, dy \, dx$$

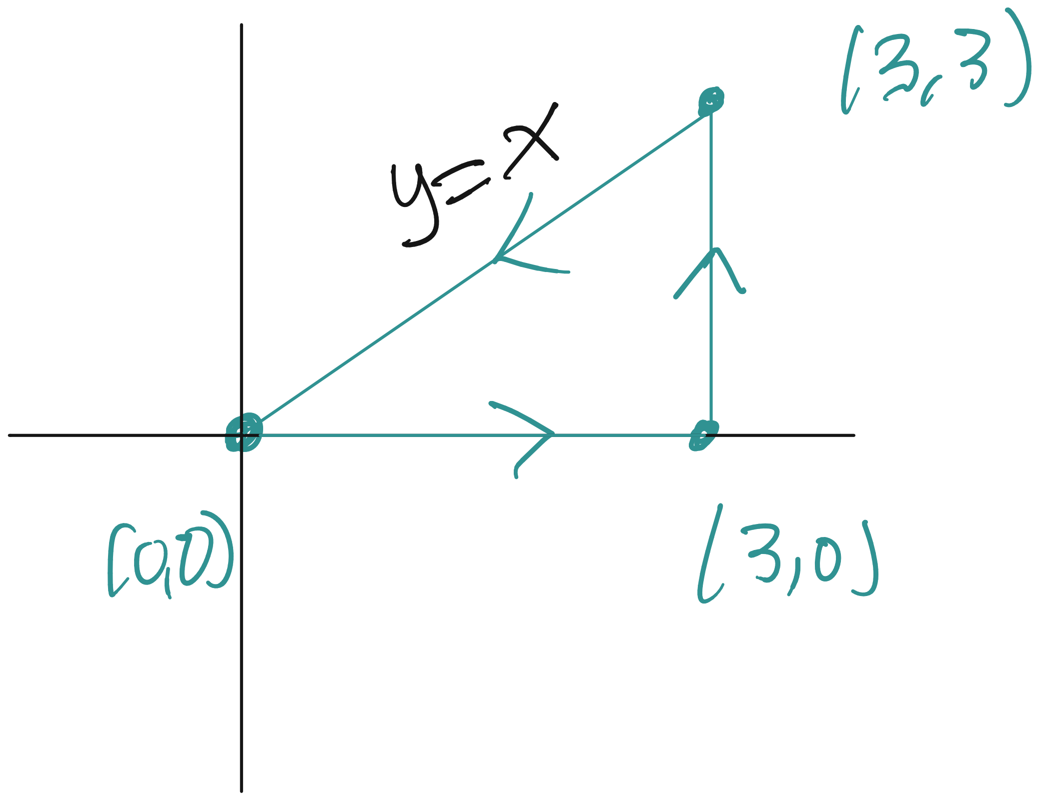
$$= \int_1^4 3xy^5 \Big|_{y=2}^{4} dx = \int_1^4 3x(1024-32) \, dx$$

$$= (1024-32) \int_1^4 3x \, dx = (1024-32)\left(\frac{3x^2}{2} \Big|_{x=1}^{4} \right)$$

$$= \boxed{(1024-32)\left(\frac{3}{2}\right)(16-1)}$$

Ex 2 $\vec{F} = \langle y^2, 3xy \rangle$

R:



$$\text{Curl}(F) = Q_x - P_y$$
$$= 3y - 2y$$
$$= y$$

Goal: Compute $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{Curl}(F)\, dA$

$$\boxed{= \int_0^3 \int_0^x y \, dy\, dx}$$

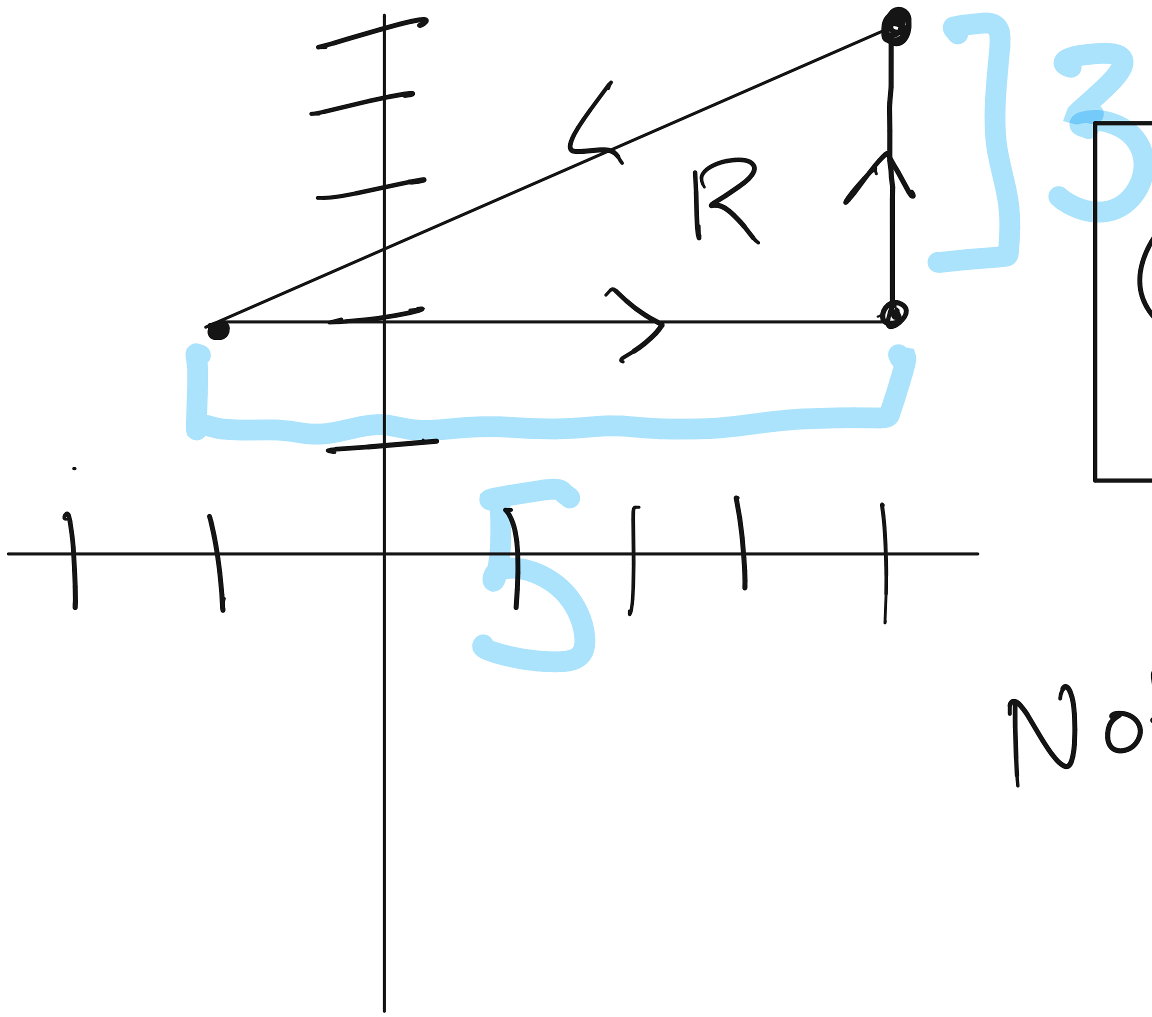
$$\int_0^3 \int_0^x y \, dy \, dx = \int_0^3 \frac{y^2}{2} \Big|_{y=0}^{x} \, dx$$

$$= \int_0^3 \frac{1}{2} x^2 \, dx = \frac{1}{6} x^3 \Big|_{x=0}^{3}$$

$$= \boxed{\frac{27}{6}}$$

Ex ③ $F = \langle \sin(x^2), 3x-y \rangle$

R right △ w/ vertices $(-1,2)$, $(4,2)$, $(4,5)$



$$\boxed{\text{Goal: } \oint_C F \cdot dr}$$

Notation $\partial R = C$ = boundary of R

$$\oint_{\partial R} \vec{F} \cdot d\vec{r}$$

$$\text{Area}(\triangle) = \frac{1}{2} \cdot 15$$

$$\mathrm{Curl}(F) = Q_x - P_y = 3 \qquad F = \langle \sin x^2,\ 3x - y \rangle$$

$$\mathrm{Curl}(F) = 3$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R 3 \, dA = 3 \cdot \boxed{\iint_R dA} = 3 \, \mathrm{Area}(\Delta)$$

$$= 3 \cdot \frac{1}{2} \cdot 15 = \boxed{\frac{45}{2}}$$

When to use Green's theorem

Requirements:

1. F needs to admit derivatives "smooth"

2. R simply connected & closed.

3. C is oriented such that R is "on the left"