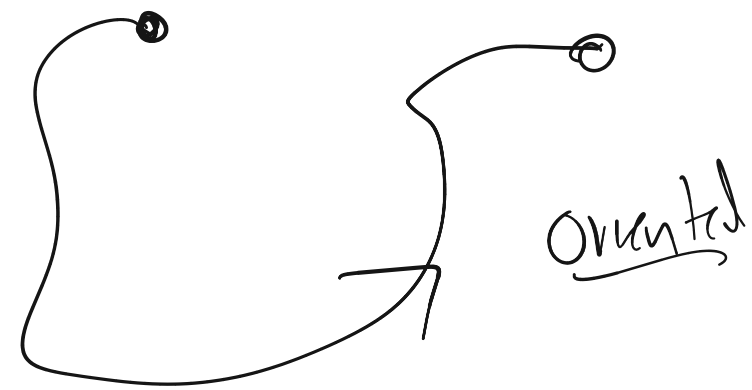
12.3 Parameterizations & line integrals (Cf also OSG 6.2)

Last time: $\vec{F}$: a vector field (2D / 3D)

C: curve (2D / 3D)

$$\int_C \vec{F}\cdot d\vec{r} = \lim_{|\Delta r_i| \to 0} \cancel{\sum_{i=0}^{n} \vec{F}(\vec{r_i}) \cdot \Delta \vec{r_i}}$$



Instead: Parameterize C by a parametric function

$$* \boxed{\vec{r}(t)} *$$

Theorem: If C is an oriented curve then there is some parametriz function $\vec{r}(t)$ whose graph is C.

Theorem: C curve, $\vec{r}(t)$ a parametrization of C $\underline{a \le t \le b}$

then: $$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

Ex $F = \langle xy, y^2 \rangle$

Start Stop

C is line segment from $(1,1)$ to $(4,3)$

① Parameterize C

$r(t)$:

$r(0) = \langle 1,1 \rangle$

$r(1) = \langle 4,3 \rangle$

$\vec{r}(t) = t\vec{v} + \vec{r}_0 = t\langle 3,2 \rangle + \langle 1,1 \rangle$

$= \langle 3t+1, 2t+1 \rangle$

$r(1) = \langle 4,3 \rangle = 1 \cdot \vec{v} + \langle 1,1 \rangle$

$\langle 3,2 \rangle = \vec{v}$

② $\int_C \vec{F} \cdot d\vec{r}$

② Compute $\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \underbrace{\vec{F}(\vec{r}(t))} \cdot \underbrace{\vec{r}'(t)}\, dt$

$F = \langle xy, y^2 \rangle$

$r = \langle 1+3t, 1+2t \rangle$

Scratch work:

ⓐ $\vec{F}(\vec{r}(t)) =$

$= \langle (1+3t)(1+2t), (1+2t)^2 \rangle$

$= \langle 1+5t+6t^2, 1+4t+4t^2 \rangle$

ⓑ $\vec{r}'(t) = \langle 3, 2 \rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^1 \underbrace{\langle 1+5t+6t^2,\ 1+4t+4t^2\rangle}_{\vec{F}(\vec{r}(t))} \cdot \overbrace{\langle 3,2\rangle}^{\vec{r}\,'(t)}\,dt$$

$$= \int_0^1 3(1+5t+6t^2) + 2(1+4t+4t^2)\,dt$$

$$= \int_0^1 5+23t+26t^2\,dt = \boxed{\frac{26}{3}+\frac{23}{2}+5}$$

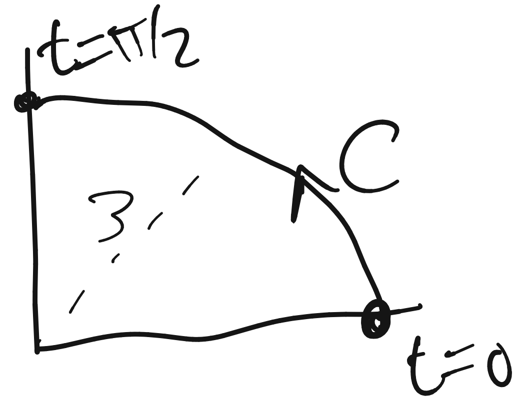
$$\vec{u} = \langle u_1, u_2 \rangle$$
$$\vec{v} = \langle v_1, v_2 \rangle$$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$$

<u>Ex</u> $F = \langle x, y^2 \rangle$

C quarter circle of radius 3 ctd @ origin contained in 1st quadrant.

$$r(t) = \langle 3\cos t, 3\sin t \rangle \quad 0 \le t \le \pi/2$$



$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \vec{F}(\vec{r}(t)) \circ \vec{r}\,'(t)\, dt \qquad F = \langle x, y^2 \rangle$$

$$\vec{F}(\vec{r}(t)) = \langle 3\cos t, 9\sin^2 t \rangle.$$

$$\vec{r}\,'(t) = \langle -3\sin t, 3\cos t \rangle$$

$$= \int_0^{\pi/2} \langle 3\cos t, 9\sin^2 t \rangle \circ \langle -3\sin t, 3\cos t \rangle \, dt.$$

$$= \int_0^{\pi/2} \left[-9 \cos t \sin t + 27 \cos t \sin^2 t \right] dt.$$

$u = \sin t$

$du = \cos t \, dt.$

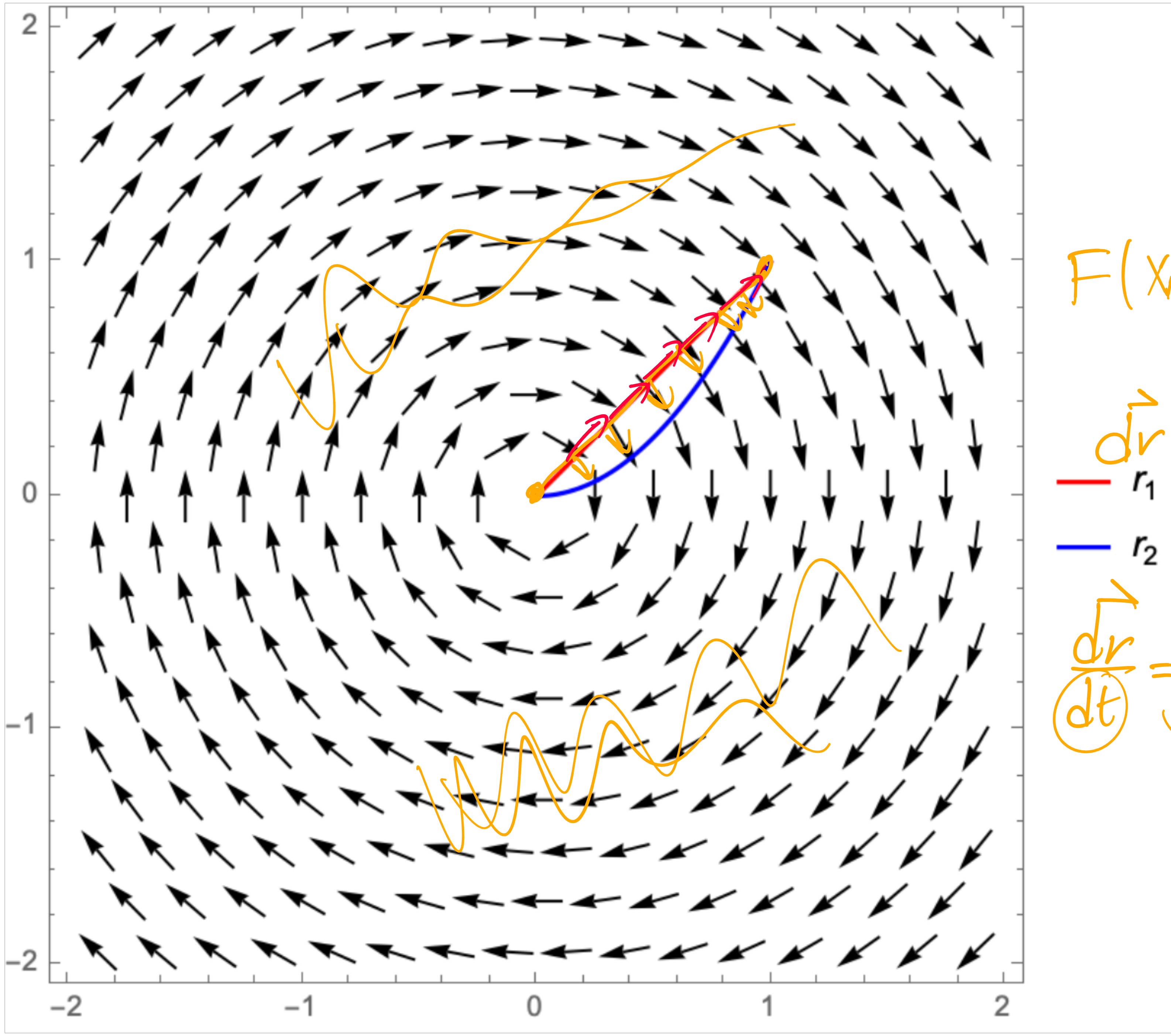
$u(0) = 0$

$u(\pi/2) = 1$

$$= \int_0^1 -9u\,du + 27u^2\,du$$

$$= \int_0^1 27u^2 - 9u \, du$$

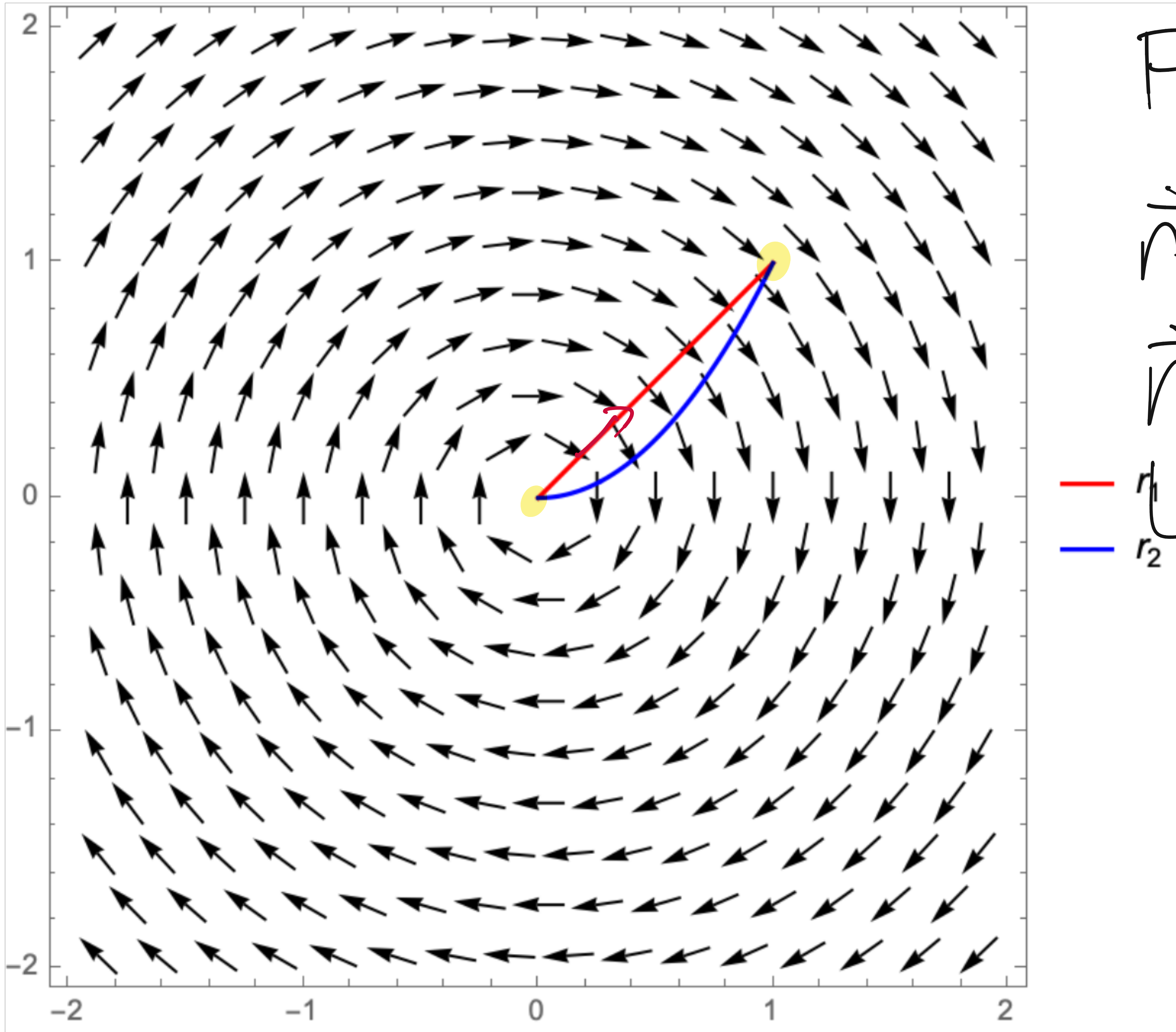
$$= \frac{27}{3} - \frac{9}{2} = 9 - \frac{9}{2} = \boxed{\frac{9}{2}}$$



$F(x,y) = F(r(t))$

$d\vec{r} = \vec{r}'(t)\,dt$

$\dfrac{d\vec{r}}{dt} = \vec{r}'(t)$



$F = \langle y, -x \rangle$

$\vec{r}_1 = \langle t, t \rangle \quad 0 \le t \le 1$

$\vec{r}_2 = \langle t, t^2 \rangle \quad 0 \le t \le 1$

① $\int_{r_1} \vec{F} \cdot d\vec{r} = \int_0^1 \langle t, -t \rangle \cdot \langle 1, 1 \rangle \, dt = \int_0^1 t - t \, dt = \boxed{0}$

② $\displaystyle\int_{r_2} \vec{F}\cdot d\vec{r} = \int_0^1 \langle t^2, -t\rangle \cdot \langle 1, 2t\rangle \, dt$

$$= \int_0^1 t^2 - 2t^2 \quad dt$$

$$= \int_0^1 -t^2 \, dt = -\frac{1}{3}t^3\Big|_0^1 = \boxed{-\frac{1}{3}}$$

different!

$F = \langle y, -x\rangle$ not path-independent.

Def: a vector field $\vec{F}$ is path-independent (aka conservative) if given any two oriented curves C_1, C_2 w/ same endpoints, we have that

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

ie. any two paths yield same numerical ans. for line integral.