12.2 The Idea of a Line Integral

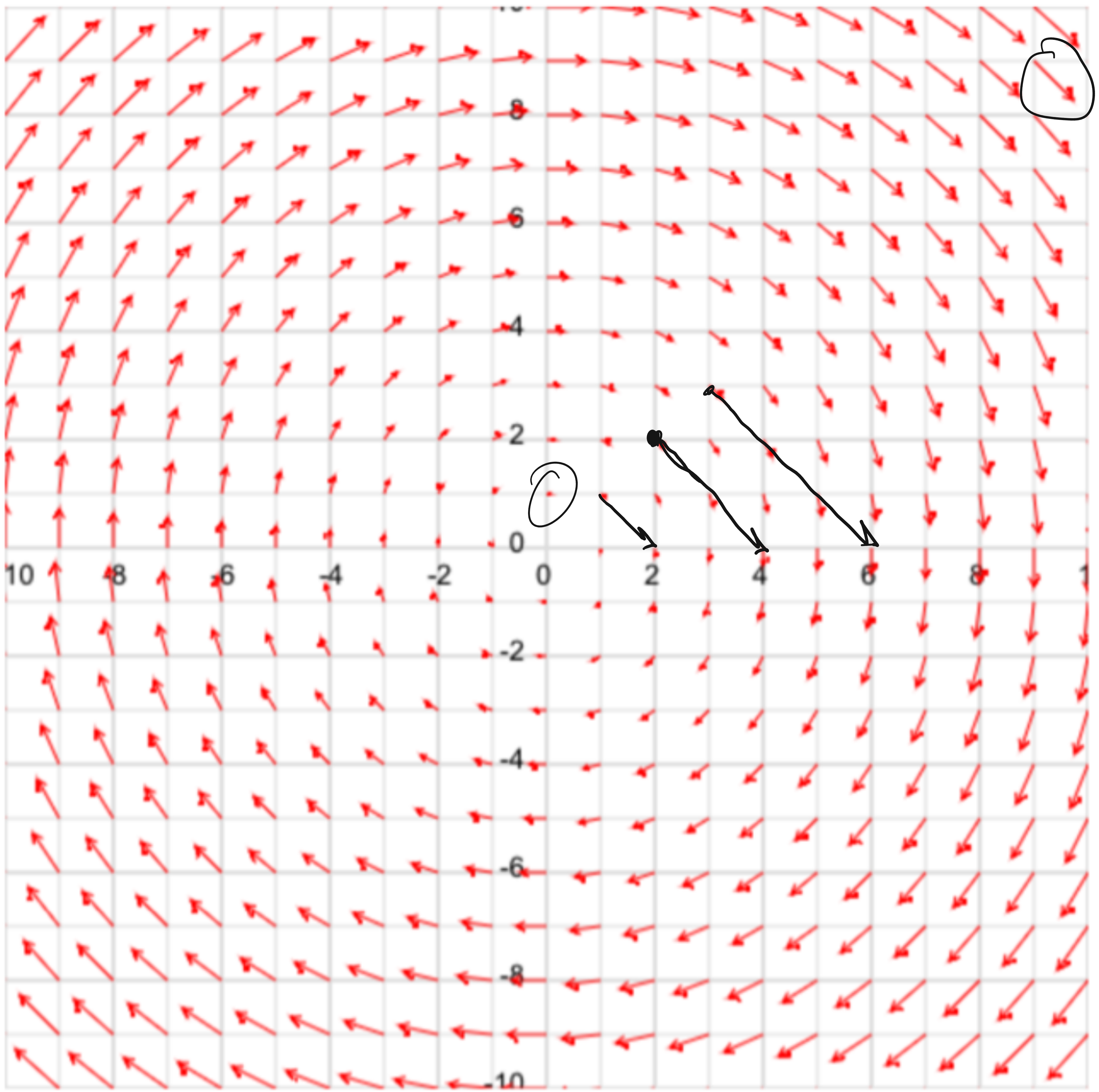
AC (Cf also 6.2 in OpenStax)

Vector field (V-field) is a function

$$\vec{F}(x,y) = \langle F_1(x,y), F_2(x,y) \rangle \text{ is}$$

a function whose inputs are points

outputs are vectors (based @ that point).



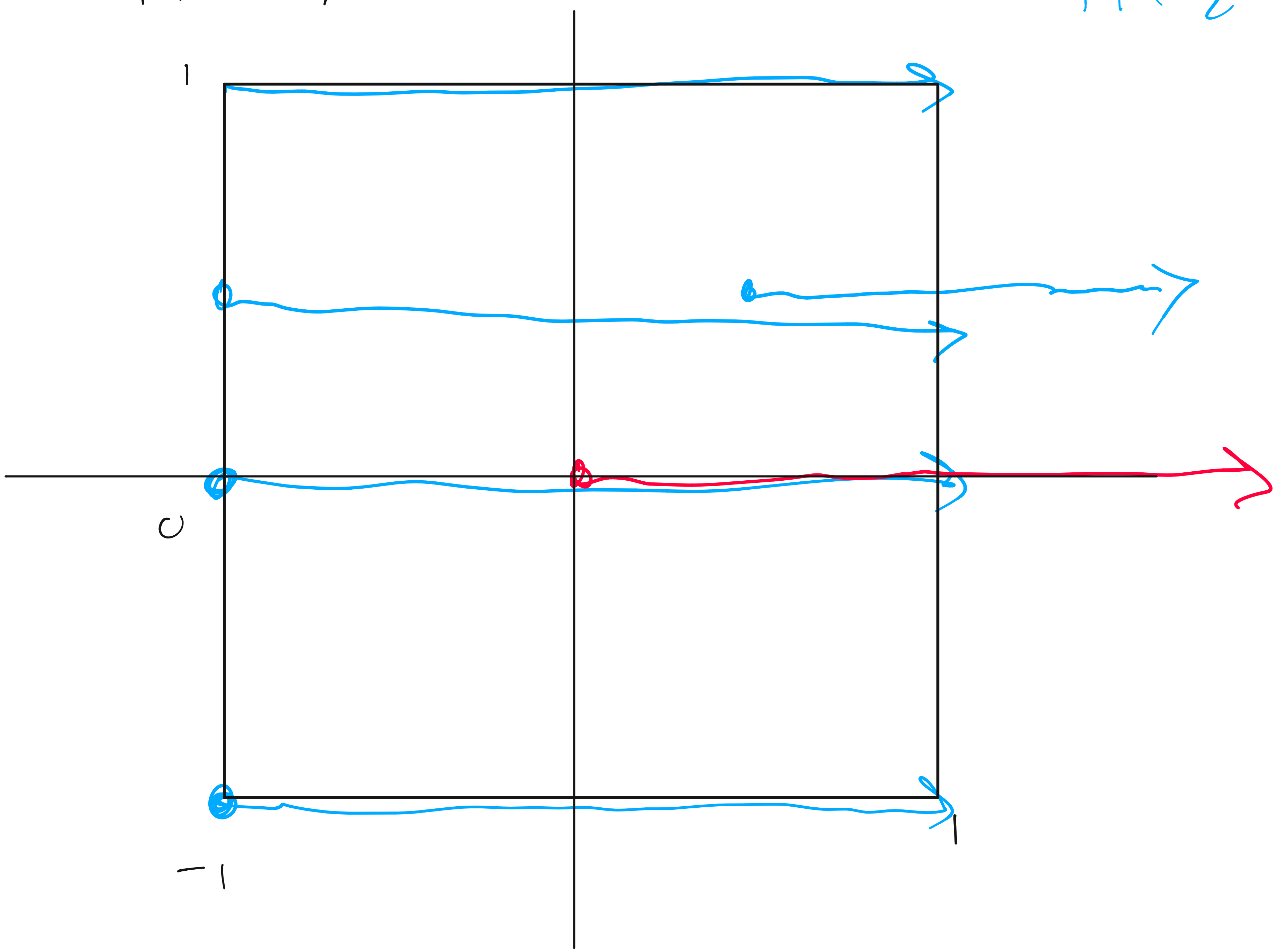
$\vec{G} = \langle y, -x \rangle$

$\langle 2, -2 \rangle$

Constant vector fields:

$\vec{H}(x,y) = \langle 2, 0 \rangle$

$H(\frac{1}{2}, \frac{1}{2}) = \langle 2, 0 \rangle$

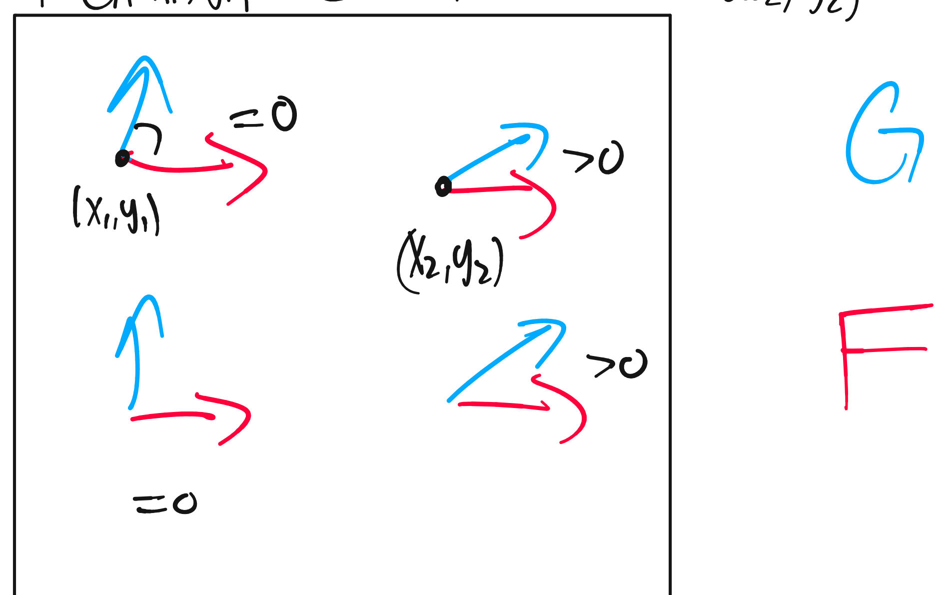


Def'n two v-fields $\vec{F}, \vec{G}$ are **orthogonal** if, for every (x,y) we have

$$\vec{F}(x,y) \bullet \vec{G}(x,y) = 0.$$

↑ dot product.

$F \cdot G(x_1,y_1) = 0$ $\quad F \cdot G > 0$ @ (x_2,y_2)



$\text{Sum}(F \cdot G) > 0$ $\qquad F \cdot G$

Ex $F = \langle x, y \rangle$

$G = \langle y, -x \rangle$

$F \cdot G = 0$: $\quad xy - yx = 0$ ✓

Gradients & Potentials

<u>Ex</u> $z = f(x,y) \Rightarrow \vec{\nabla} f = \langle f_x(x,y), f_y(x,y) \rangle$

is a <u>vector field</u>

$$\text{Scalar} \;\underset{\text{(sometimes) potential}}{\overset{\text{gradient}}{\rightleftarrows}}\; \text{Vector fields}$$

$\vec{F}(x,y)$ v-fld a potential function for $\vec{F}(x,y)$ is a scalar func $f(x,y)$ st $\vec{\nabla} f = \vec{F}$

Notes/ Problems:

① Not every $\vec{F}(x,y)$ admits a potential function f.

② How do we determine if $\vec{F}(x,y)$ admits a scalar potential $f(x,y)$?

③ Given an $\vec{F}$ that we know has a potential func, how do we find a potential func for $\vec{F}$?

Ex $\vec{F}(x,y) = \langle x, y \rangle$ admits a scalar potential.

Find a $f(x,y)$ st $\vec{\nabla} f = \vec{F}$.

such that

$g_x(y) = 0$

$\frac{\partial f}{\partial x} = x \Rightarrow$ guess $f(x,y) = \frac{1}{2}x^2 + g(y)$

$f(x,y)$ make a guess @ $f(x,y)$

guess: $f(x,y) = \frac{1}{2}x^2 + g(y)$ $\Rightarrow$ $f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + C$

$\partial f/\partial y$

take $\frac{\partial f}{\partial y} = F_2 = y$

check:

$\vec{\nabla} f = \langle x, y \rangle = \vec{F}$ ✓ :)

$g'(y) = y$

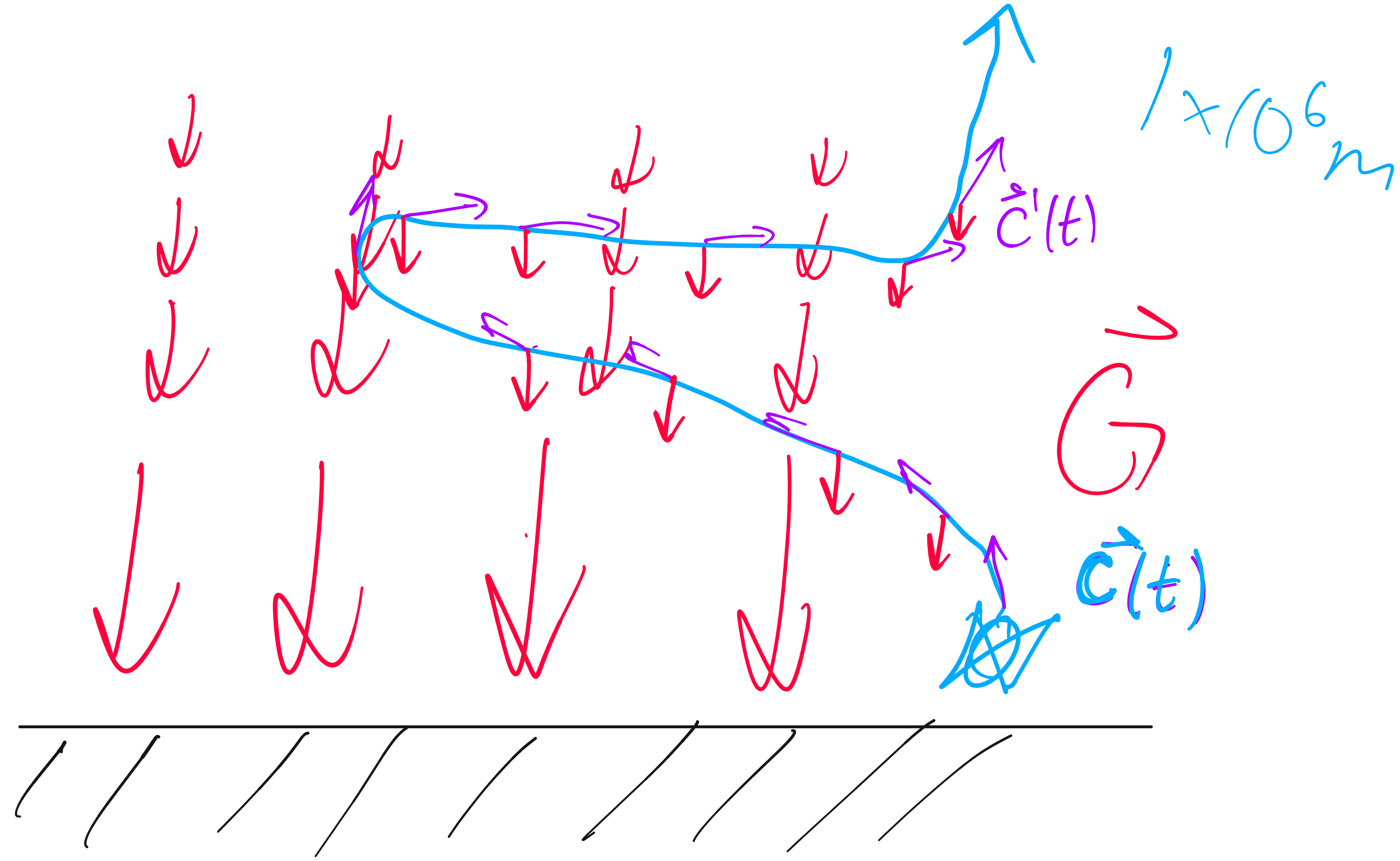
$F = \langle x, y \rangle$

$g(y) = \frac{1}{2}y^2 + C$

Motivation:

Consider Gravitational field:

$$G(x,y) = -\frac{GMm}{(R+y)^2}\hat{j}$$



R = radius of Earth.

$$\text{Work} = \sum(\vec{G}\cdot\vec{c}'(t))\,\Delta t$$

$$W = \vec{F}\cdot\overrightarrow{\text{displacement}}$$

Line integral: ① Vector field $\vec{F}(x, y, \dots)$

② (Parametric) Curve $\vec{C} = \vec{C}(t)$

$d\vec{r}$ = displacement vector.

$$\int_C \vec{F} \cdot d\vec{r} = \lim_{|\Delta r| \to 0} \sum_{i=0}^{n} F(\vec{r}_i) \cdot \vec{\Delta r}_i$$

integrate over curve C.

Open Stax:

$$\vec{F} = \langle P, Q \rangle$$

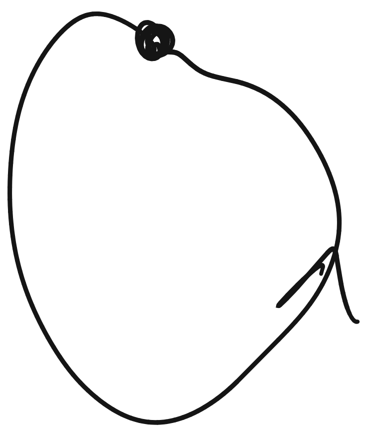
$$d\vec{r} = \langle dx, dy \rangle$$

old-fashioned

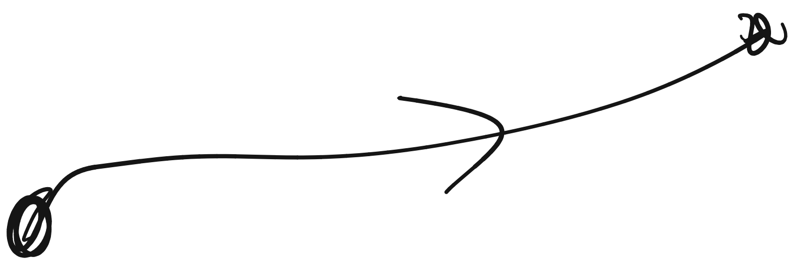
$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy$$

OS not'n.

Curve C is closed if it starts & ends @ same point.



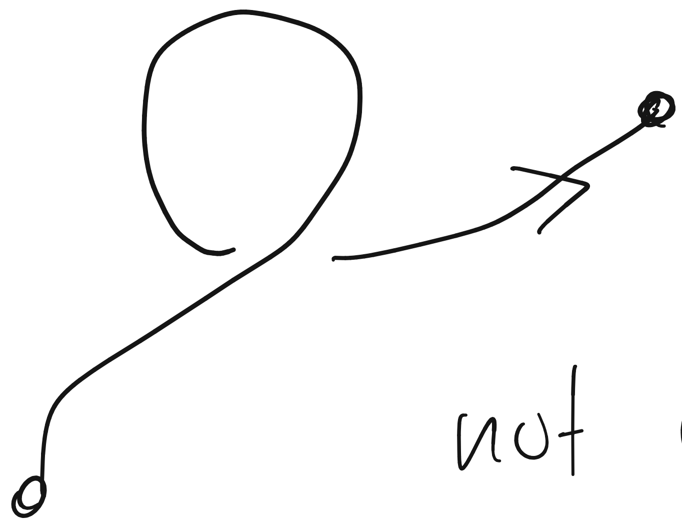
closed curve



not closed

In 3D:

closed.



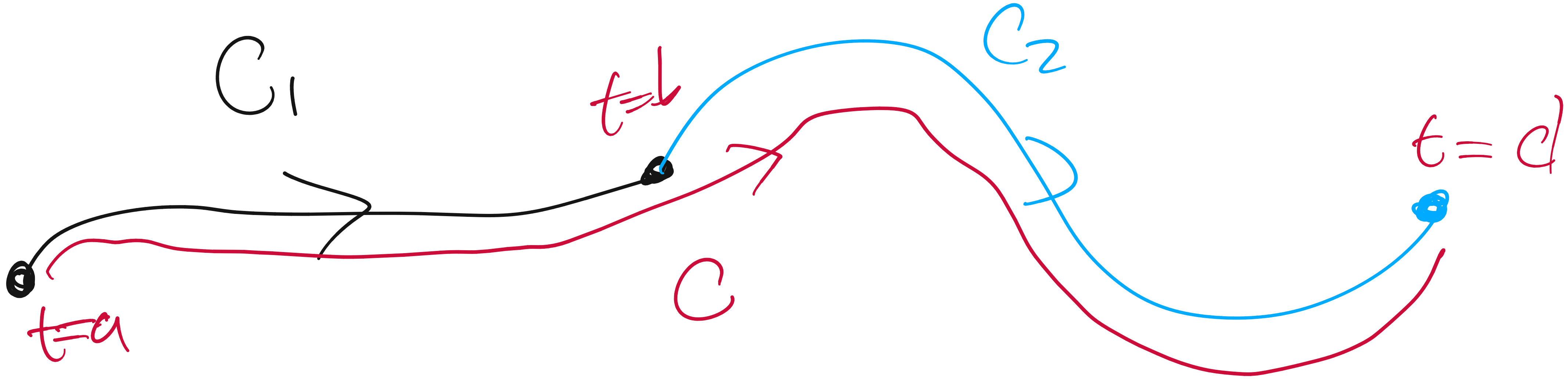
not closed.

If C is a closed curve

Write $\oint_C \vec{F} \cdot d\vec{r}$ rather than $\int_C \vec{F} \cdot d\vec{r}$

Means same thing, but closed curves are
(special.

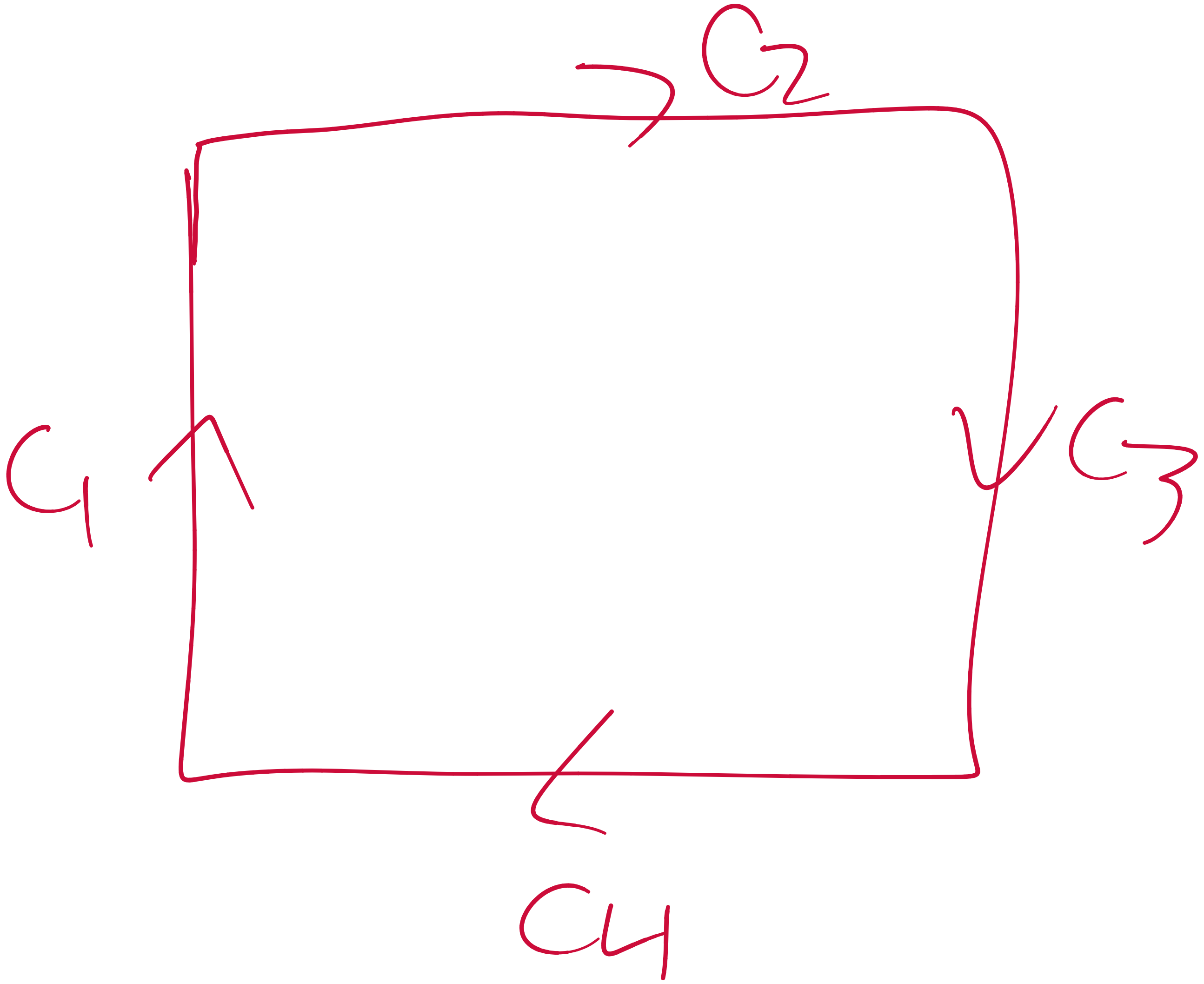
Ex



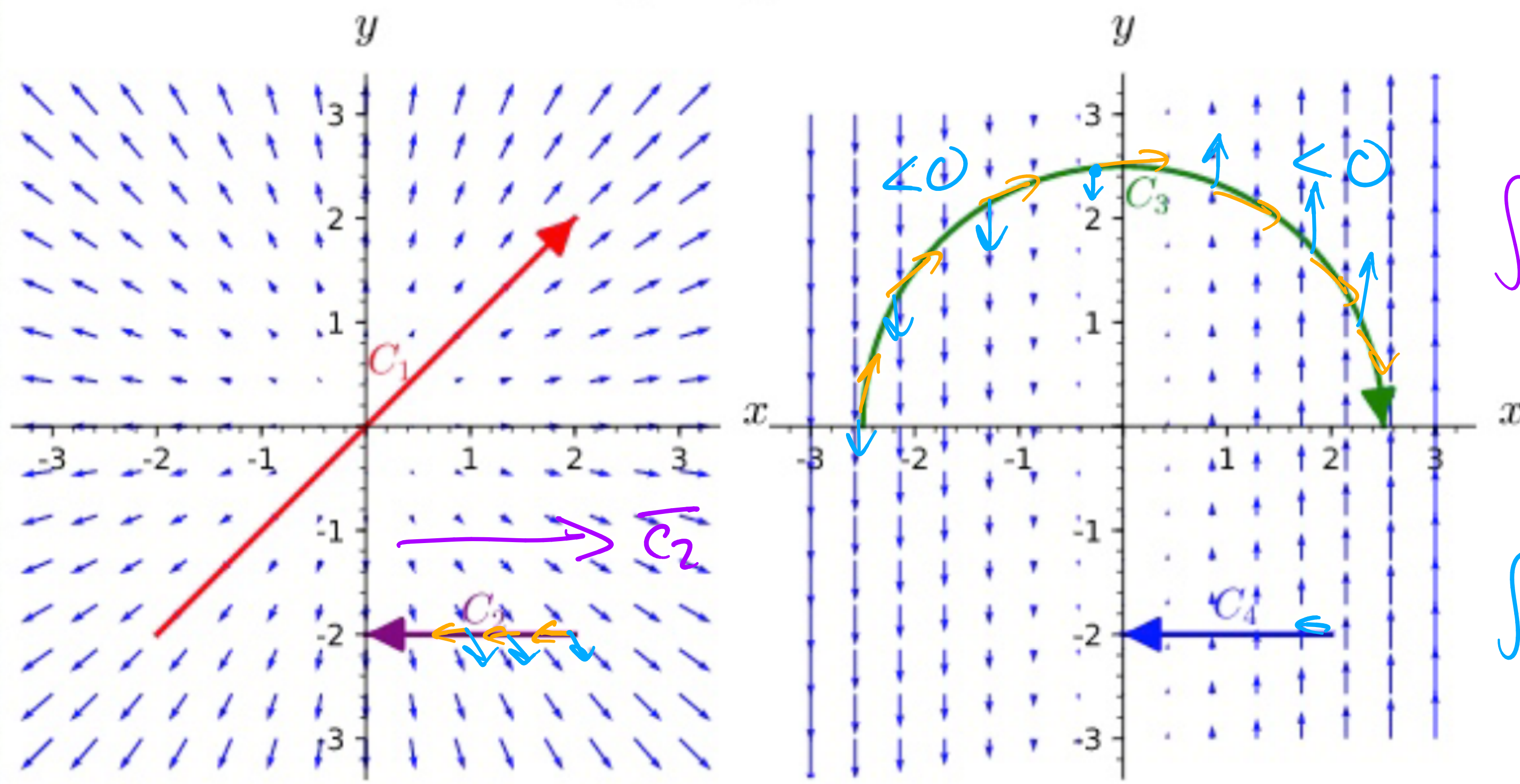
$C = C_1 + C_2$

$$C(t) = \begin{cases} C_1(t) & ; \quad a \le t \le b \\ C_2(t) & ; \quad b \le t \le d \end{cases}$$

$$\int_{C_1+C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$



C_1
C_2
C_3
C_4



all · are neg.

(a) $\vec{F}$ **(b)** **G**

$$\int_{C_3} \vec{G}\cdot\vec{dr} < 0$$

$$\int_{C_4} \vec{G}\cdot\vec{dr} = 0$$

Figure 12.2.4. Vector fields and oriented curves

(a) $\displaystyle\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = 0$

(b) $\displaystyle\int_{C_2} \mathbf{F}\cdot d\mathbf{r} < 0$

$$\int_{\overline{C_2}} \vec{F}\cdot\vec{dr} > 0$$

(c) $\displaystyle\int_{C_3} \mathbf{G}\cdot d\mathbf{r}$

(d) $\displaystyle\int_{C_4} \mathbf{G}\cdot d\mathbf{r}$