§9.7: Derivatives & Integrals of V-V Functions

In Calc I: $s(t)$ is a (scalar-valued) function

$$s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

The same def'n works for v-v functions:

$$\frac{d}{dt}\left(\vec{r}(t)\right) := \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$= \lim_{h \to 0} \frac{\langle x(t+h) - x(t), y(t+h) - y(t), z(t+h) - z(t) \rangle}{h}$$

$$= \lim_{h \to 0} \left\langle \frac{x(t+h) - x(t)}{h}, \ldots \right\rangle$$

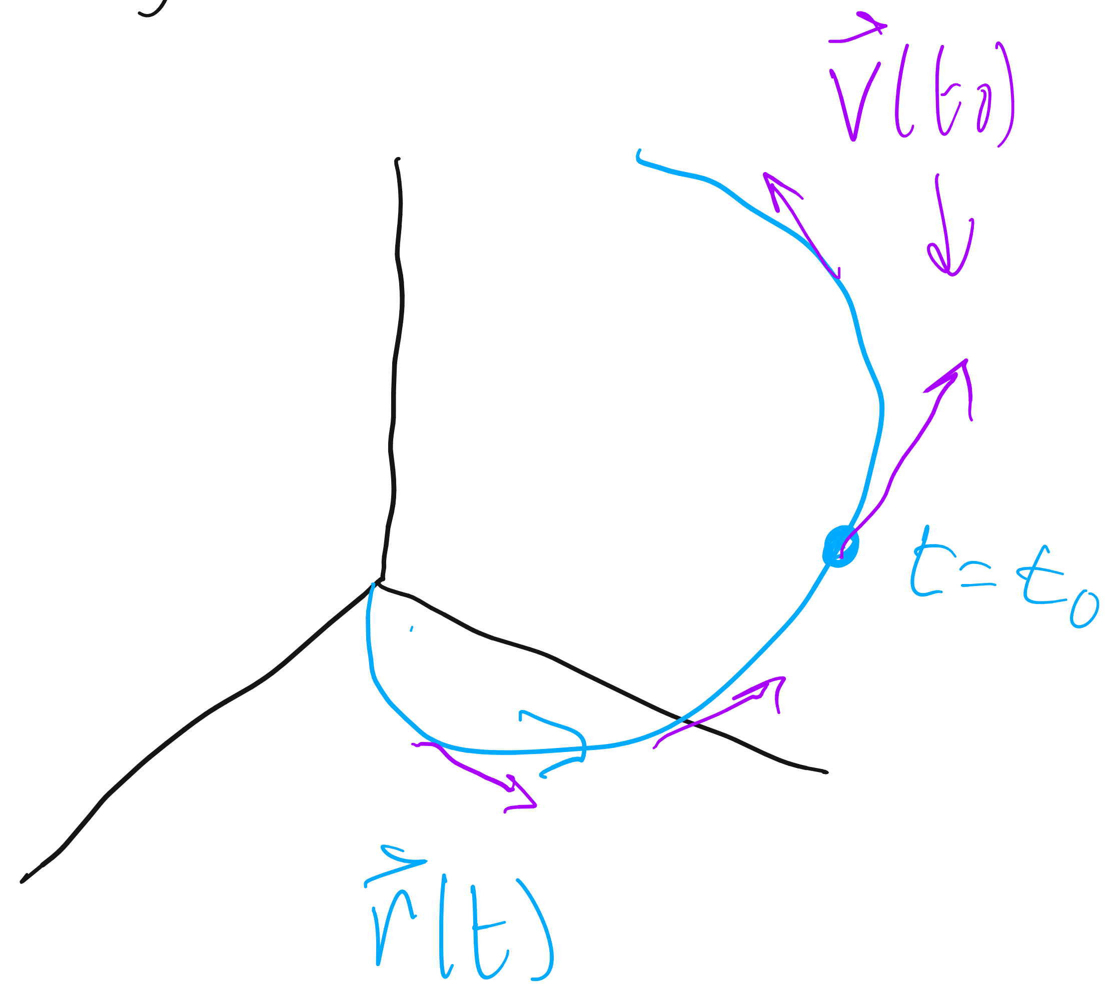
$$= \langle x'(t), y'(t), z'(t) \rangle.$$

<u>So:</u> $\vec{r}'(t) = \langle x', y', z' \rangle$ <u>"Component-wise" derivative</u>

we interpret $\vec{r}\,'(t)$ as the instantaneous velocity of our motion @ the $t=t$.

$\vec{r}\,''$: acceleration

$\vec{v} = \vec{r}\,'$, $\vec{a}(t) = \vec{r}\,''(t)$



Ex for each of the following find $r'(t) =: \vec{v}(t)$

① $\langle \cos t, t\sin t, \ln t \rangle$ $\qquad t\cos t + \sin t$

$\langle -\sin t, t\cos t + \sin t, \frac{1}{t} \rangle = \vec{r}'(t)$

② $\langle t^2+3t, e^{-2t}, t^2+1 \rangle$

$\Downarrow$

$\langle 2t+3, -2e^{-2t}, 2t \rangle = \vec{r}'(t)$

Facts about derivs of V-V. functions:

$\vec{f}(t)$, $\vec{g}(t)$, Scalar function $s(t)$

1. $$\boxed{\frac{d}{dt}(\vec{f} \pm \vec{g}) = \vec{f}' \pm \vec{g}'.}$$

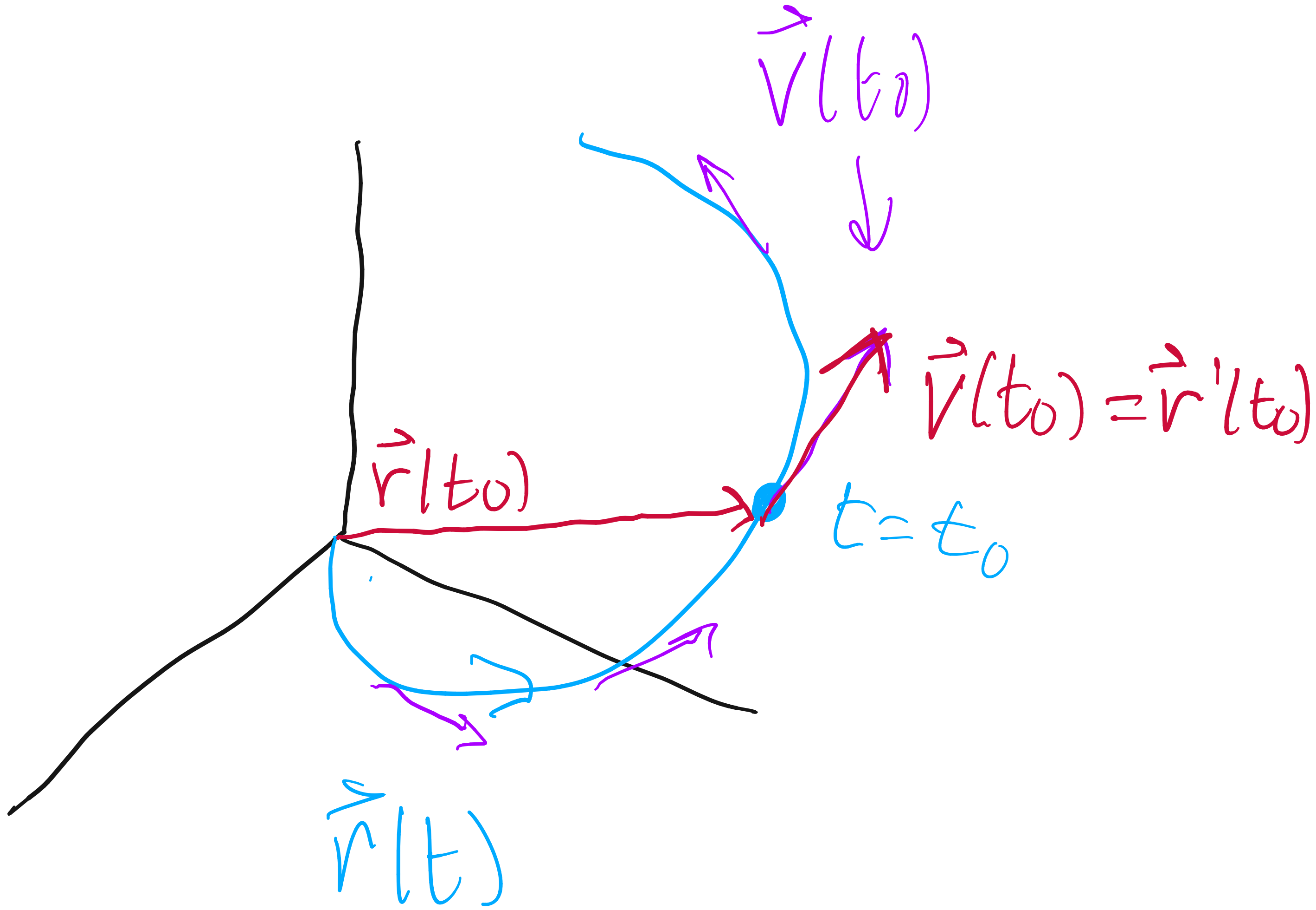
2. $$\frac{d}{dt}(\vec{f} \cdot \vec{g}) = \vec{f}' \cdot \vec{g} + \vec{f} \cdot \vec{g}'.$$

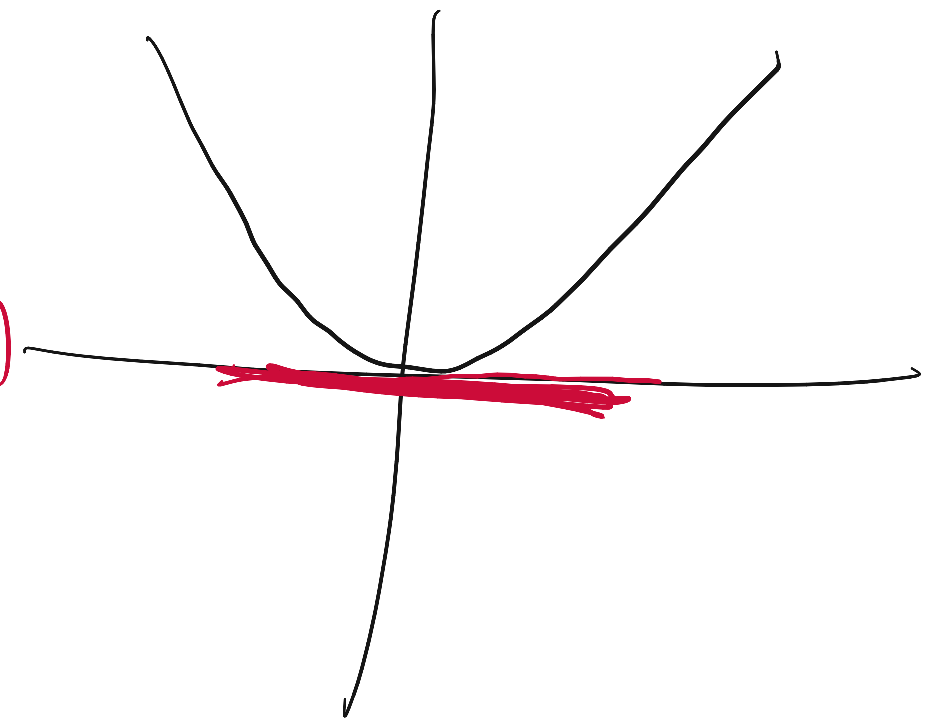
3. $$\frac{d}{dt}(\vec{f} \times \vec{g}) = \vec{f}' \times \vec{g} + \vec{f} \times \vec{g}'$$ Order matters!

4. $$\frac{d}{dt}(s(t)\vec{f}(t)) = s'(t)\vec{f}(t) + s(t)\vec{f}'(t).$$

Product rule

5 $\frac{d}{dt}\left(\vec{f}(s(t))\right) = s'(t)\,\vec{f}'(s(t))$ (chain rule)





$$L(x)= f'(x_0)(x-x_0)+f(x_0)$$

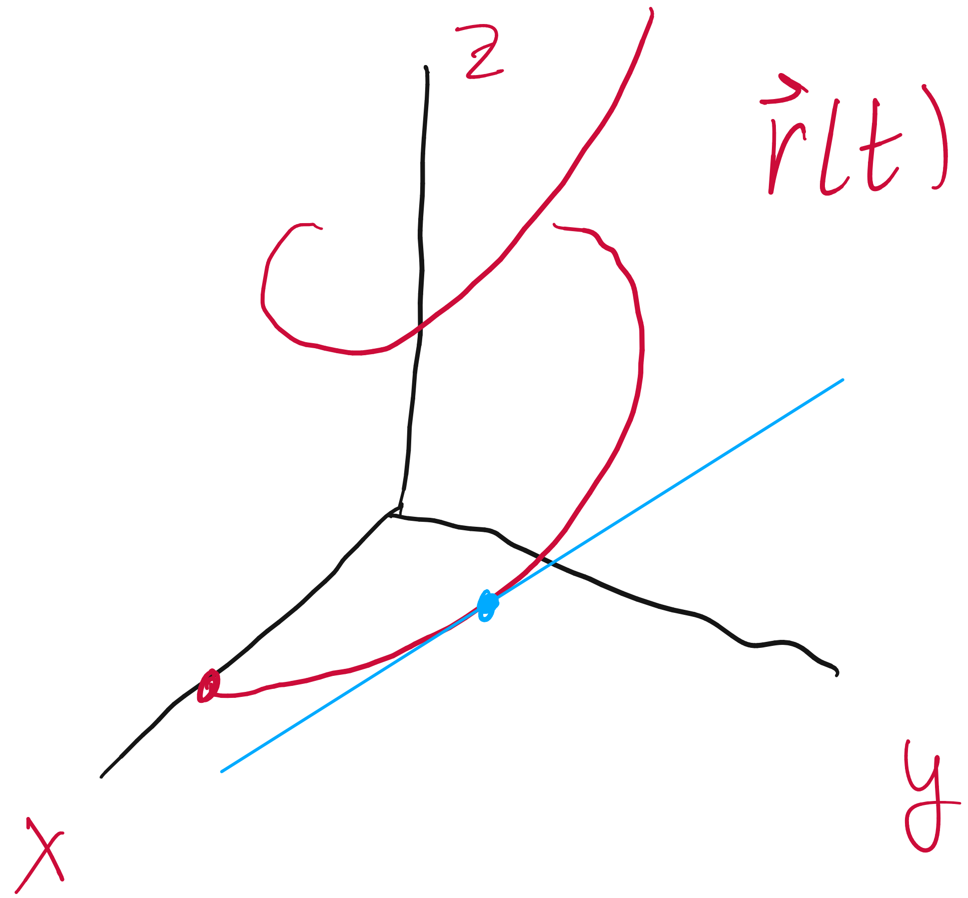
have start point : $\vec{r}(t_0)$

start dir : $\vec{r}\,'(t_0)$

$$\boxed{\vec{L}(t) = \vec{r}(t_0) + t\,\vec{r}\,'(t_0)}$$

Eqn of tangent line to $\vec{r}(t)$ @ $t=t_0$

Ex if $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$



find tangent line @

the $t = \pi/4$

① St. point : $\vec{r}(\pi/4) = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \pi/4 \rangle$

② St. direction.

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\vec{r}\,'(\pi/4) = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 \rangle$$

$\vec{L}(t) = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4} \right\rangle + t\left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 \right\rangle.$

Integrals of V-V functions:

$\vec{f}(t)$ a v.v. function an anti<u>derivative</u> of $\vec{f}(t)$ is a function $\vec{F}(t)$ such that

$$\vec{F}'(t) = \vec{f}(t).$$

$$\vec{C} = \langle C_1, C_2, C_3 \rangle$$

$$\int \vec{f}(t)\,dt = \vec{F}(t) + \vec{C}$$

Integration is done component-wise ie:

if $\vec{f} = \langle x, y, z \rangle$ (all funcs of t)

$$\int \vec{f}\,dt = \left\langle \int x\,dt, \int y\,dt, \int z\,dt \right\rangle$$

Ex $\vec{v}(t) = \langle -2\sin 2t,\ 2\cos t,\ t \rangle$.

find an $\vec{r}(t)$ st $\vec{r}(0) = \langle 1, 0, 0 \rangle$.

$(\vec{v}(t) = \vec{r}\,'(t))$

$$\int \vec{v}(t)\,dt = \vec{r}(t) + \vec{C}$$

$r_x = \int -2\sin 2t \, dt = \cos 2t + C_1$

X-comp. of r.

$r_y = \int 2\cos t \, dt = 2\sin t + C_2$

$r_z = \int t \, dt = \frac{t^2}{2} + C_3$

$\vec{r} = \langle \cos 2t + C_1, \; 2\sin t + C_2, \; \frac{t^2}{2} + C_3 \rangle.$

Want $\vec{r}(0) = \langle 1, 0, 0 \rangle.$

$r_x \quad r_y \quad r_z$

$\vec{r}(0) = \langle 1, 0, 0 \rangle = \langle 1 + C_1, \; 0 + C_2, \; 0 + C_3 \rangle$

$$1 = 1 + c_1$$
$$0 = 0 + c_2 \Rightarrow c_1 = c_2 = c_3 = 0.$$
$$0 = 0 + c_3$$

definite integrals also work!!

$$\int_a^b \vec{f}(t)\,dt = \vec{F}(b) - \vec{F}(a)$$

where $\vec{F}$ is an anti-deriv of $\vec{f}$.

ie FTC works!