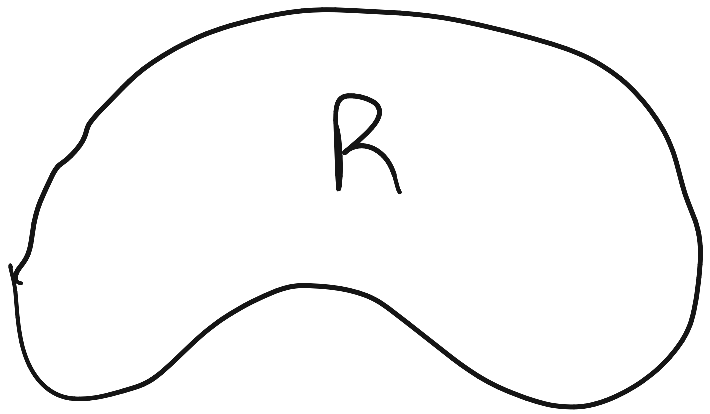
11.7 Triple Integrals

Double integrals tells us about mass/area of a 2D region

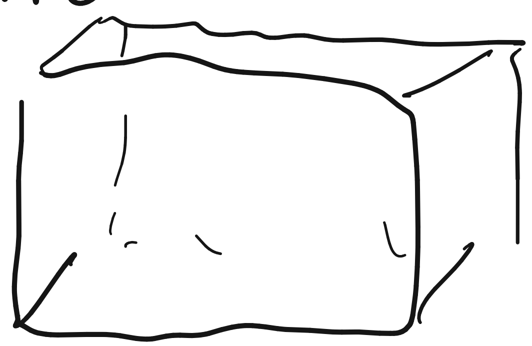
region $R \subseteq \mathbb{R}^2$

"lives inside".



$$\iint_R \_\_ \, dA$$

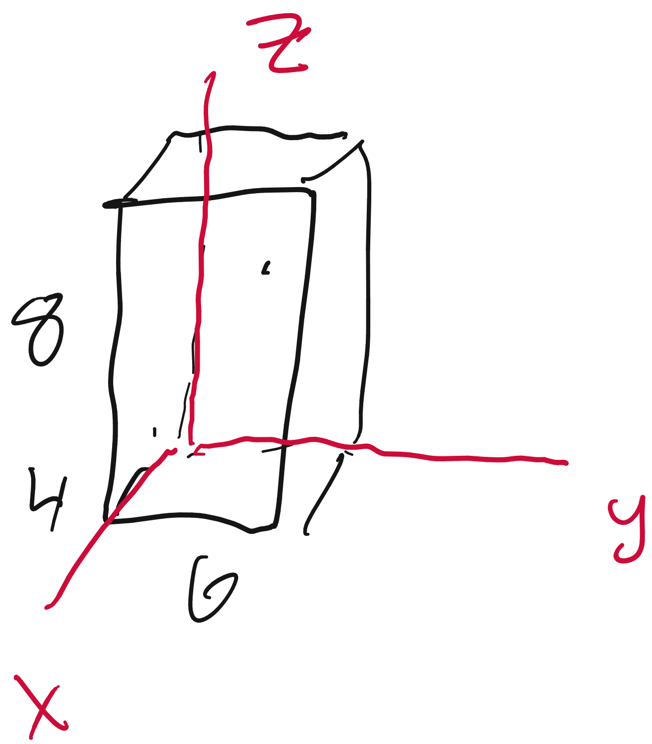
triple integrals are much the same, but in 3D.



$S \subseteq \mathbb{R}^3$

Ex Solid block of marble,

Its dimensions are $4m \times 6m \times 8m$



Goal: determine the mass of this block given a density function $\delta(x,y,z)$.

Chop into small cubes & add up masses.

$$M = \iiint_B \delta(x,y,z)\, dV$$

Quantities computable w/ triple integrals

① **Mass**: $\delta(x,y,z)$ density function

$S \subseteq \mathbb{R}^3$ a region.

↑
"Solid"

$dV = dx\,dy\,dz$ or some reordering of this.

$$M = \iiint_S \delta(x,y,z)\,dV$$

② **Volume**: $S \subseteq \mathbb{R}^3$ region

$$\mathrm{Vol}(S) = \iiint_S 1\,dV$$

③ Avg Value: $S \subseteq \mathbb{R}^3$ region

"lives in" ie is a subset of.

f a function defined on S.

think

$$\boxed{\text{Avg}_S(f) = \frac{1}{\text{Vol}(S)} \cdot \iiint_S f(x,y,z)\, dV.}$$

avg value of f over S

$\left(\text{Compare w/ 2D: } \text{Avg}_R(f) = \frac{1}{\text{Area}(R)} \iint_R f\, dA. \right)$

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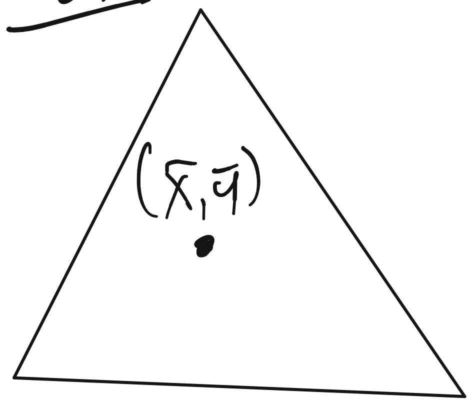
④ Centers of Mass: S a region in $\mathbb{R}^3$, δ density function

$(\bar{x}, \bar{y}, \bar{z})$

$M = \iiint_S \delta\, dV$ total mass.

$\bar{x} = \frac{1}{M} \iiint_S x \cdot \delta\, dV$, similar for $\bar{y}, \bar{z}$.

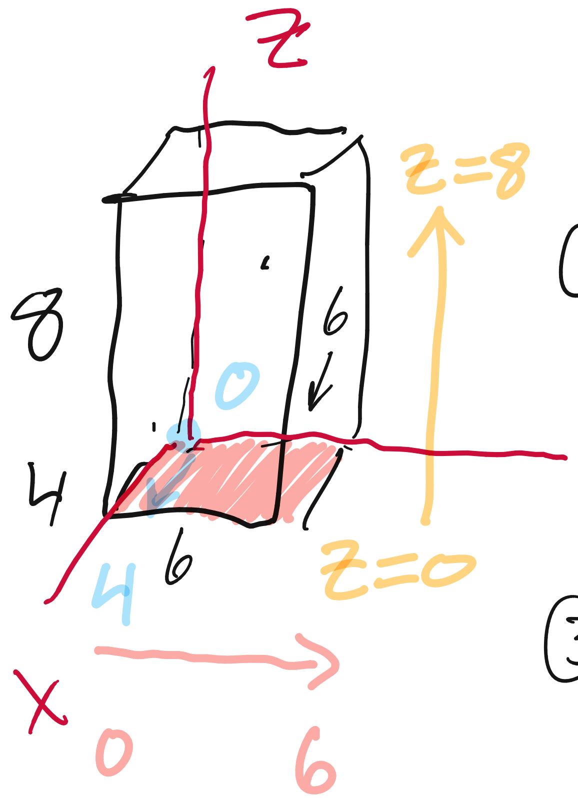
2D Ex



$\delta = 1$

$\bar{x}, \bar{y}$ represent the avg of the x, y coords over this triangle, Δ.

Return to Motivating example:



$\delta = x$

$\delta = x.$

① $0 \le x \le 4$
② $0 \le y \le 6$
③ $0 \le z \le 8.$

$$\int_{z=0}^{8} \int_{y=0}^{6} \int_{x=0}^{4} x \, dx\,dy\,dz.$$

$M = 384$ Kg

Side bar!
Fubini's theorem applies here b/c we're over a cuboid.

$$\int_{z=0}^{8}\int_{y=0}^{6}\underbrace{\int_{x=0}^{4} x \, dx}\, dy\, dz = \int_0^8\int_0^6 \frac{1}{2}x^2\Big|_{x=0}^{4} \, dy\, dz$$

$$= \int_0^8\int_0^6 \frac{16}{2} - \frac{0}{2} \, dy\, dz$$

$$= \int_0^8\int_0^6 8 \, dy\, dz$$

$$= \int_0^8 8y\Big|_{y=0}^{6} \, dz$$

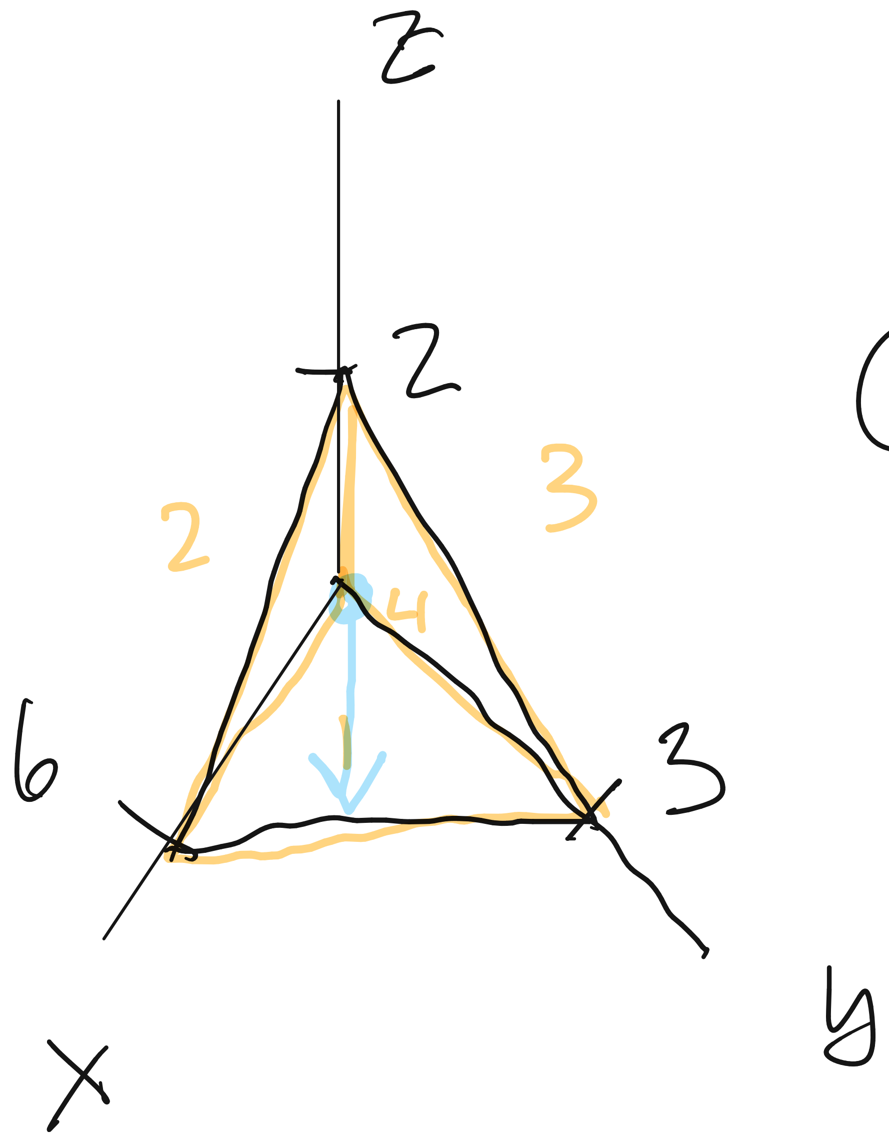
$$= \int_0^8 48 - 0 \, dz = \int_0^8 48 \, dz$$

$$= 48z\Big|_0^8 = 48 \cdot 8 = \boxed{384}$$

Ex find mass of the tetrahedron bounded by the coord. planes and the plane

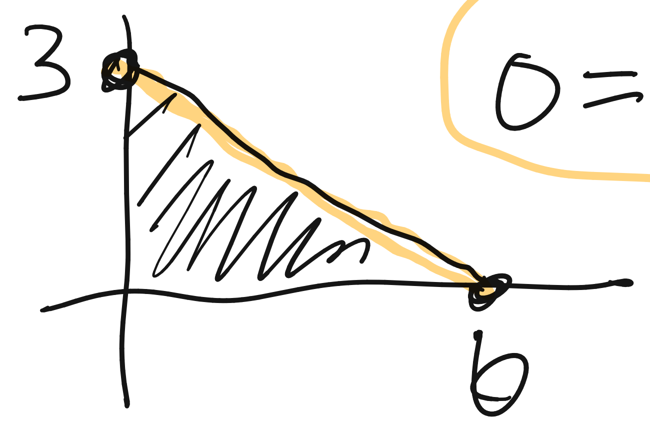
$z = \frac{1}{3}(6 - x - 2y)$ | $\delta(x,y,z) = x+y+z$

$0 \le z \le \frac{1}{3}(6-x-2y)$



Q: what about x, y bounds?

look @ z=0 slice



$0 = \frac{1}{3}(6-x-2y)$ ①

↓ Solve for y in terms of x

Work out relationship btwn x, y. $y = 3 - \frac{1}{2}x$

(2) $0 \le y \le 3 - \frac{1}{2}x$

(3) $0 \le x \le 6$.

$$\int_0^6 \int_0^{3-\frac{1}{2}x} \int_0^{\frac{1}{3}(6-x-2y)} x+y+z \; dz \, dy \, dx$$

$$= \int_0^6 \int_0^{3-\frac{1}{2}x} xz + yz + \frac{1}{2}z^2 \Big|_{z=0}^{z=\frac{1}{3}(6-x-2y)} dy \, dx$$

$$= \cdots = \int_0^6 \int_0^{3-\frac{1}{2}x} \frac{4}{3}x - \frac{5}{18}x^2 - \frac{7}{9}xy + \frac{2}{3}y - \frac{4}{9}y^2 + 2 \, dy\, dx$$

$$= \cdots = \int_0^6 5 + \frac{1}{2}x - \frac{7}{2}x^2 + \frac{13}{216}x^3 \, dx = \frac{33}{2}$$