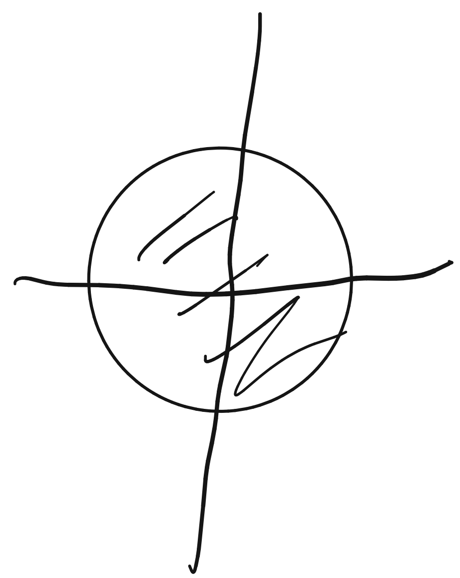
Double Integrals in Polar Coords (§11.5)

<u>Ex</u> (Motivation)

$f(x,y) = e^{x^2+y^2}$ Integrate this on unit disk

$D: \; x^2+y^2 \le 1.$

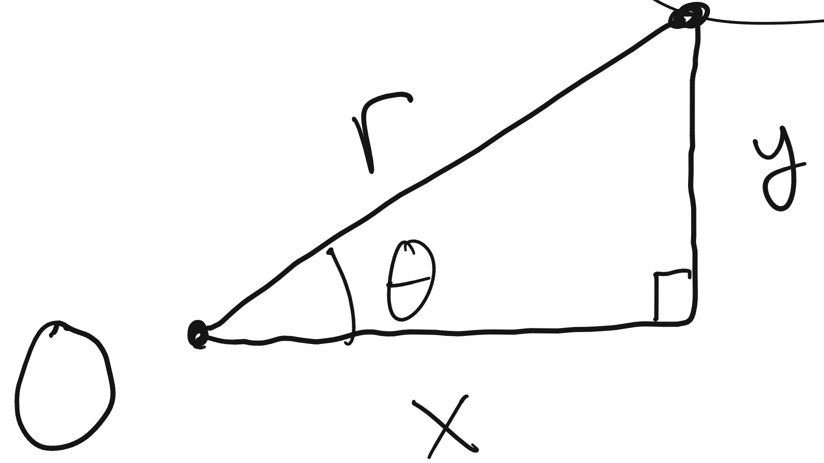


$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{x^2+y^2} \, dy\,dx$$

has no anti-deriv expressible in terms of "Elementary funcs."

New tool: Polar Coords:

$P = (x, y)$



describe the same point.

In polar Coords: $P = (r, \theta)$

note: need to be careful about domain/range

to convert:

Rect. Coords. → polar Coords

$$(x, y) \longmapsto r^2 = x^2 + y^2, \quad \theta = \tan^{-1}(y/x)$$

Polar Coords

$(r,\theta) \mapsto$ $x = r\cos\theta \quad , \quad y = r\sin\theta$

$(x,y) = (3,4) \longrightarrow (r,\theta)$

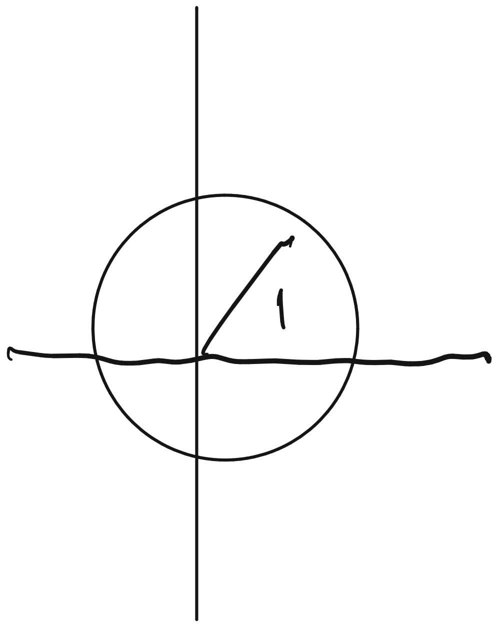
$$r^2 = 3^2 + 4^2 = 9 + 16 = 25$$

$$r^2 = 25, \rightarrow \boxed{r = 5}$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$

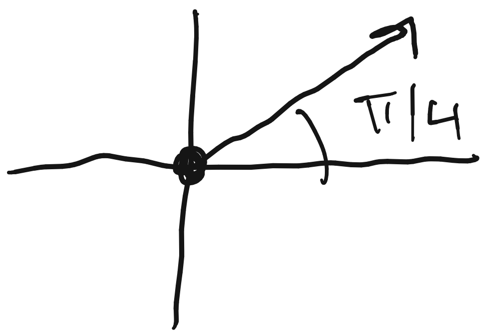
Regions in Polar coords:

① What does the curve $r=1$ look like?

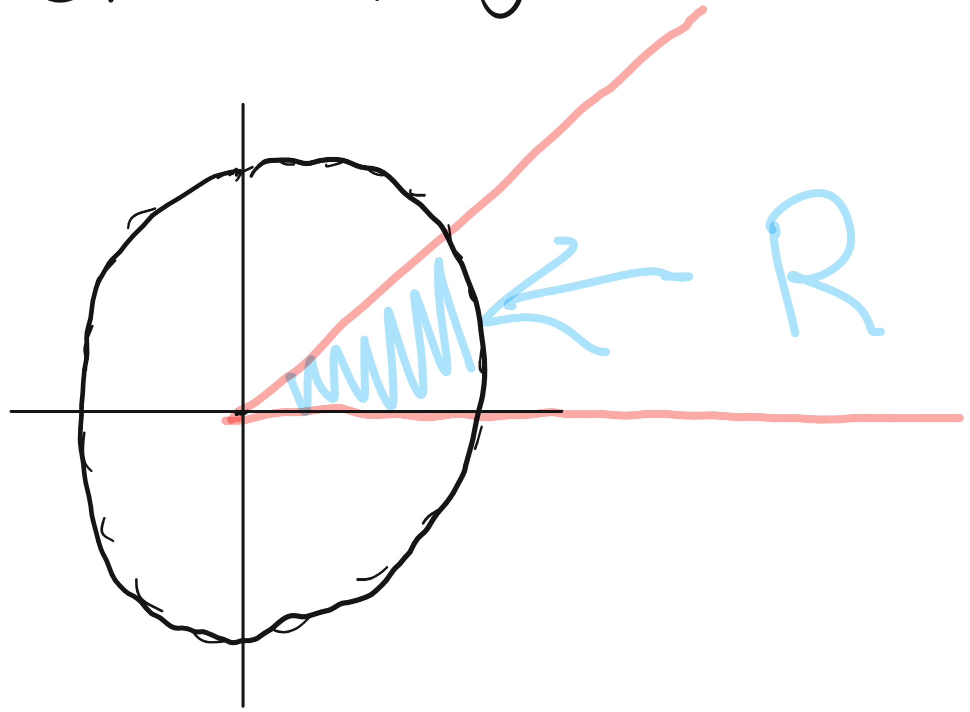


$r=1 \Rightarrow r^2=1 \Rightarrow x^2+y^2=1$

② what does curve $\theta=\pi/4$ look like?



③ Draw region $0 \le r \le 1, \quad 0 \le \theta \le \pi/4$



this is a "rectangle" in polar coords

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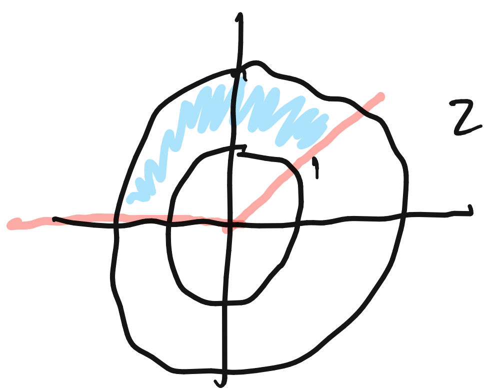
④ $1 \le r \le 2, \quad \pi/4 \le \theta \le \pi$



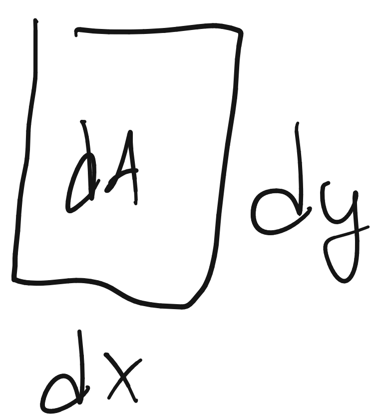
this is ALSO a rectangle!

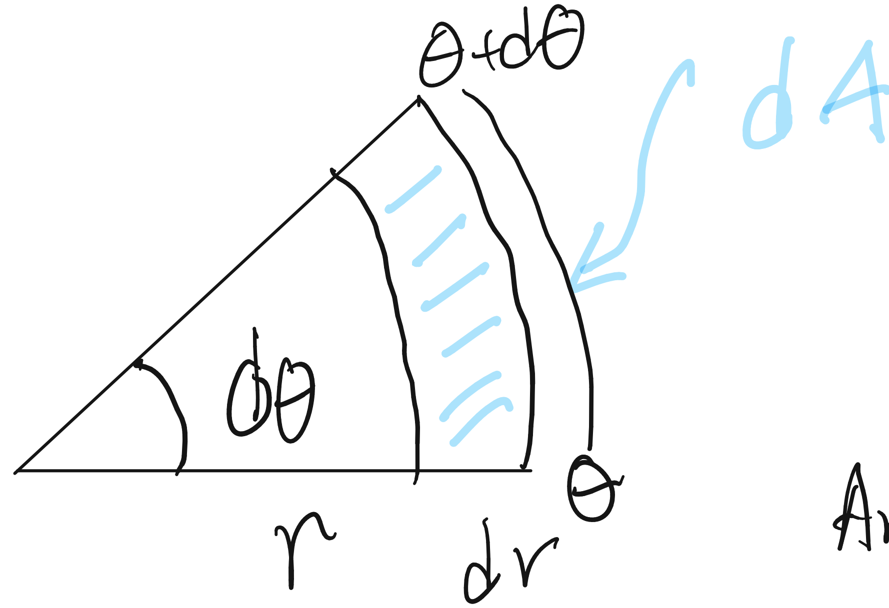
Integrals in Polar coords:

$$f(x,y) \longmapsto f(r,\theta)$$

this is done by Substituting $x = r\cos\theta$,
$y = r\sin\theta$ into f.

Hard / Open part: What happens to dA?



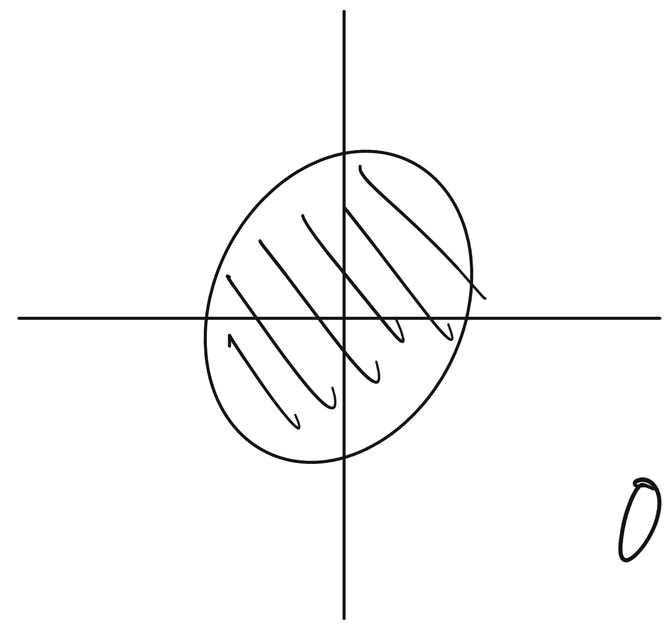


width $= dr$

length $= r\,d\theta$

Area $=$ $\boxed{dA = r\,d\theta\,dr \text{ or } r\,dr\,d\theta}$

Ex $f(x,y) = e^{x^2+y^2}$ $D = \{ x^2+y^2 \leq 1 \}$



$f(r,\theta) = e^{r^2}$

$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

$f(r,\theta) = e^{r^2\cos^2\theta + r^2\sin^2\theta} = e^{r^2}$

rectangle! $\Rightarrow$

Fubini applies!

$$\iint_D e^{r^2}\, dA$$

$$\int_0^1 \int_0^{2\pi} e^{r^2}\, r\, d\theta\, dr = \int_0^1 r\theta e^{r^2} \Big|_{\theta=0}^{2\pi} dr$$

$$= 2\pi \int_0^1 r e^{r^2}\, dr$$

Do a U-Sub!

$$u = r^2 \rightarrow du = 2r\,dr$$

$$= \pi \int_0^1 e^u\, du = \pi \cdot (e^1 - e^0) = \boxed{\pi(e-1)}.$$

$$\iint_D e^{x^2+y^2}\,dA = \pi(e-1).$$