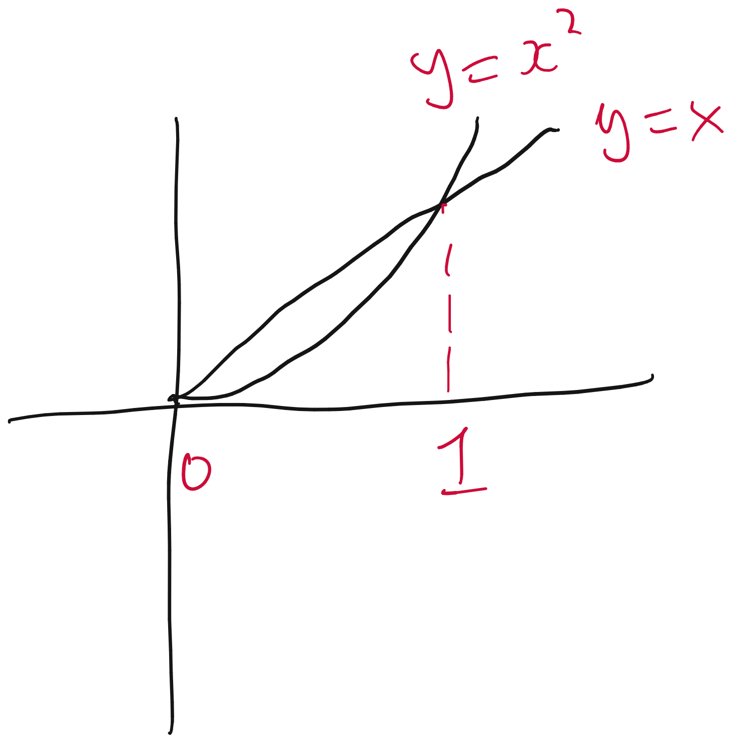
Last time: [§11.4: Applications of Double integrals]

Double integrals → How to set them up & how to evaluate them!



inner integral can contain functions of the outside variable.

$$\underbrace{\int_{\_}^{\_} \int_{\_}^{\_}}_{\text{bounds}} f(x,y)\, dA$$

dA: $dy\,dx$ or $dx\,dy$

$$\int_0^1 \int_{x^2}^{x} f(x,y)\, dy\, dx$$

Today: Applications & interpretation.

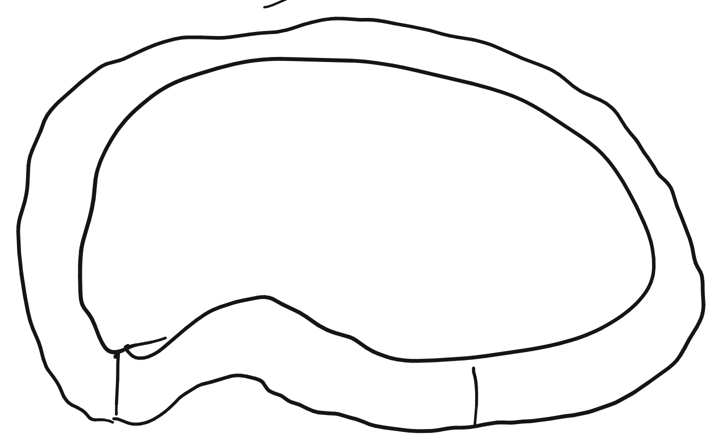
Application 1: Mass

Lamina "thin plate"

Setup: Region R lives in plane

units:

density function $\delta(x,y)$ $\left[\mathrm{kg}/\mathrm{m}^2 \right]$

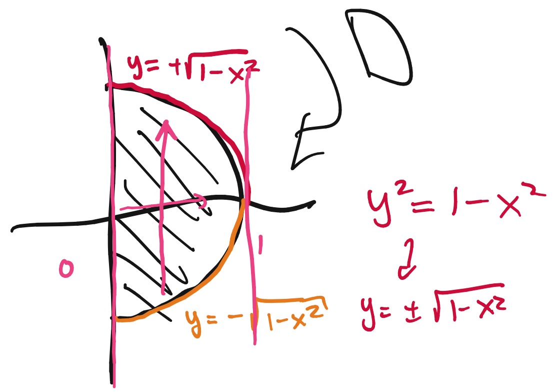


Mass of R is just

think: height = density

$$M = \iint_R \underbrace{\delta(x,y)}_{\mathrm{kg}/\mathrm{m}^2} \, \underbrace{dA}_{\mathrm{m}^2}$$

Ex D is the right half of the unit disk. ctd @ origin.



$$\delta(x,y) = x$$

Goal: find mass of D given our density function.

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} x \, dy\, dx$$

$$= \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\, dy\, dx = \int_0^1 xy \Big|_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}} dx$$

$$= \int_0^1 2x\sqrt{1-x^2}\,dx = \int_1^0 -\sqrt{u}\,du = -\int_0^1 \sqrt{u}\,du$$

$u = 1 - x^2$

$du = -2x\,dx$

$u(1) = 0, \quad u(0) = 1$

$$= -\frac{2}{3} u^{3/2} \Big|_0^1$$

$$= \boxed{\frac{2}{3} \text{ kg}}$$

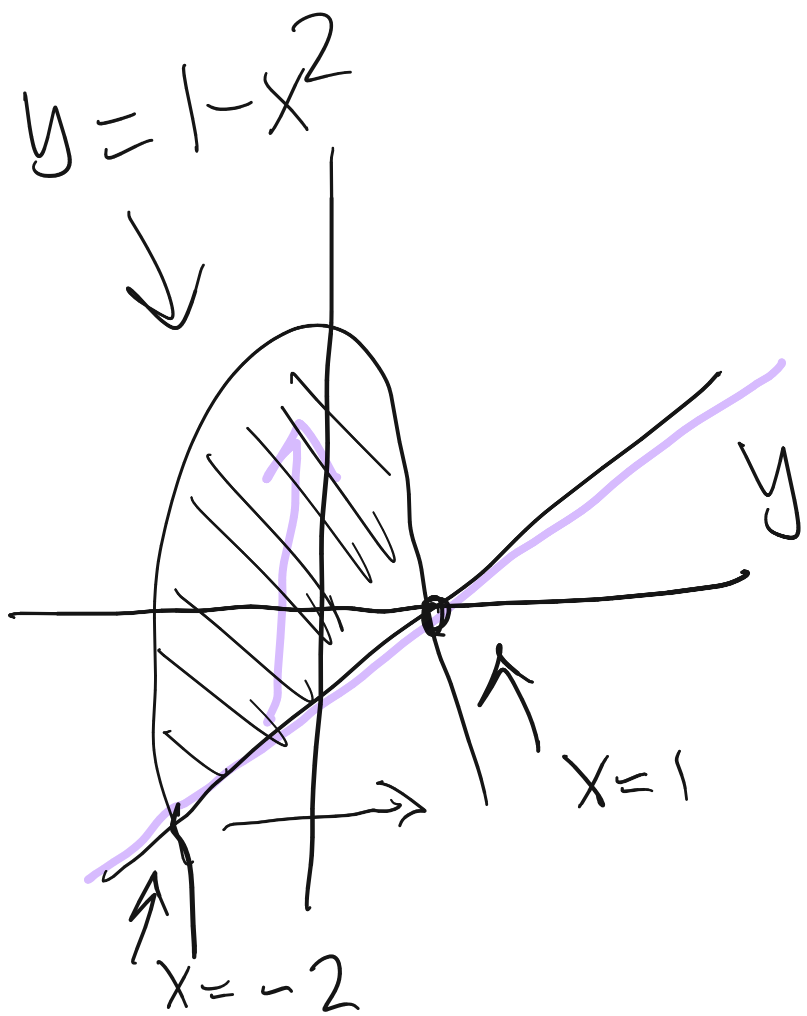
(Recall: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$)

App. 2: Area:

R a region in the plane,

(no units)

then $\text{Area}(R) = \iint_R 1 \, dA$



$1 \cdot \text{Area}(R)$

Calc I/II way:

$$\int_a^b [f(x)-g(x)] \, dx$$

$$\int_{-2}^{1} \int_{x-1}^{1-x^2} 1 \, dy \, dx$$

$$= \int_{-2}^{1} y \Big|_{y=x-1}^{y=1-x^2} dx = \int_{-2}^{1} (1-x^2) - (x-1) \, dx$$

$$= \int_{-2}^{1} 2 - x - x^2 \, dx = \boxed{4.5}$$

App. 3 Center of Mass / (Moment of inertia)

Setup: density function $\delta(x,y)$

Region R in plane.

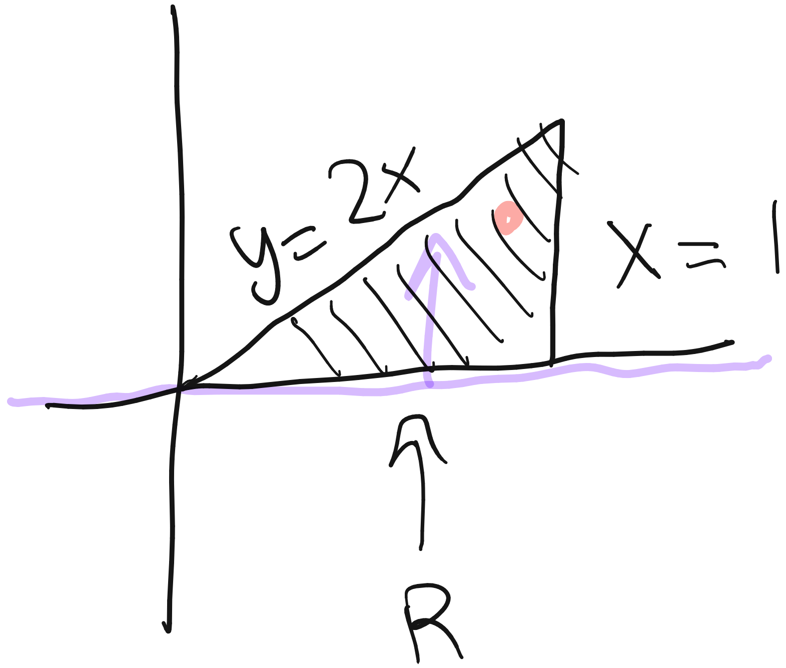
find average loc'n of mass in x, y directions sep'ly

① find total mass $M = \iint_R \delta(x,y)\, dA$

② CM is the point $(\bar{x}, \bar{y})$ where:

$$\bar{x} = \frac{1}{M} \iint_R x \cdot \delta(x,y)\, dA, \quad \bar{y} = \frac{1}{M} \iint_R y\, \delta(x,y)\, dA$$

Ex



$$\delta(x,y) = 6x + 6y + 6$$

① find $M = \iint_R \delta(x,y)\,dA$

$$= \int_0^1 \int_0^{2x} (6x+6y+6)\,dy\,dx = 14 \text{ Kg}$$

② find $\bar{x}$, $\bar{y}$

$$\bar{X} = \frac{1}{M} \cdot \iint_R \overset{m}{x}\, \overset{kg/m^2}{\delta(x,y)}\, \overset{m^2}{dA}$$

kg

$$= \frac{1}{M} \cdot \int_0^1 \int_0^{2x} x \cdot \overbrace{(6x+6y+6)}^{\delta(x,y)}\, dy\, dx$$

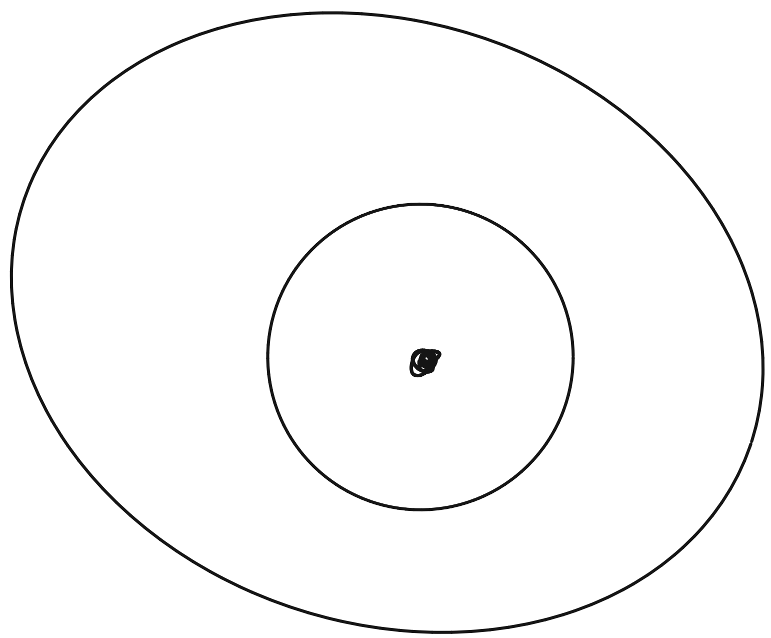
y

$$= \frac{1}{14} \int_0^1 \int_0^{2x} 6x^2 + 6xy + 6x \, dy\, dx = \boxed{5/7}\ m$$

$$\bar{y} = \frac{1}{M} \iint_R y\,\delta(x,y)\, dA = \frac{11}{14}\ m$$

$$CM = \left(5/7,\ 11/14\right)$$ = point.

Caution: Center of mass need not lie w/in that region!



$\delta(x,y) = 1$