§11.2 Iterated Integrals

Yesterday, we saw that

$$\iint_R f(x,y)\,dA \approx \frac{\min + \max}{2}$$

Today, we'll see how to <u>actually</u> compute a double integral "analytically"

Recall A slice of a function is a curve of the form $z = f(a,y)$ or $z = f(x,b)$ for some fixed value(s) a, b.

Idea: first integral computes the Area of a slice

2^{nd} integral adds up all of the slice-areas.

Ex $f(x,y) = 25 - x^2 - y^2$

find volume below this surface when

$-3 \le x \le 3, \quad -4 \le y \le 4$

Idea: fix x-value $x_0 \in [-3, 3]$.

$$A(x_0) := \int_{-4}^{4} f(x_0, y)\, dy$$

$\uparrow$

Slice & find area.

$f = 25 - x^2 - y^2$

$$A(x_0) = \int_{-4}^{4} 25 - x_0^2 - y^2 \, dy$$

Constant!

$$= (25 - x_0^2)y - \frac{1}{3}y^3 \Big|_{y=-4}^{4}$$

$$= 200 - \frac{128}{3} - 8x_0^2$$

$\leftarrow$ Area of the Slice $f(x_0, y)$

$$\underbrace{\int_{-3}^{3} \int_{-4}^{4} 25 - x^2 - y^2 \, dy}\, dx = 800$$

① Compute inner integral

② Compute outer integral.

$$\int_{-3}^{3} A(x)\, dx = \int_{-3}^{3} \left(200 - 8x^2 - \frac{128}{3}\right) dx = \boxed{800}$$

$f(x,y) = 25 - x^2 - y^2$ here

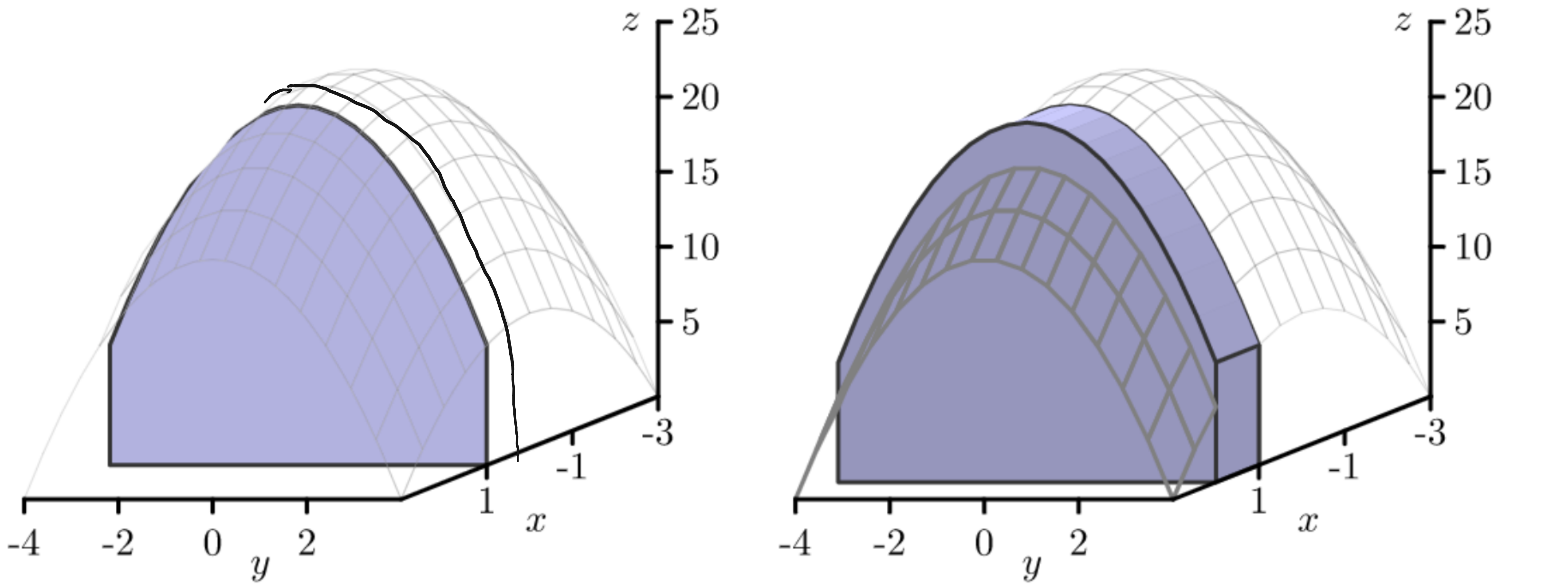

Figure 11.2.2. Left: A cross section with fixed x. Right: A cross section with fixed x and Δx.

Another Ex:

$f(x,y) = 3+x+y$

Our Estimate from yesterday was that

$$\int_0^2 \int_0^2 f(x,y)\,dx\,dy \approx 20$$

$$\int_0^2 \left[\int_0^2 3+x+y \quad dx \right] dy =$$

$$\int_0^2 \left(3x + \frac{x^2}{2} + xy \Big|_{x=0}^{2} \right) dy$$

$$= \int_0^2 \left(6 + \frac{4}{2} + 2y \right) dy = \int_0^2 (8+2y)\,dy$$

Theorem (Fubini's Theorem)

R is a rectangle

$a \le x \le b, \quad c \le y \le d$ AND

if $f(x,y)$ is Cts on R then,

$$\iint_R f(x,y)\,dA =$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$= 8y + y^2 \Big|_{y=0}^{2}$$

$$= 16 + 4 - 0 = \boxed{20}$$

happy coincidence!

Ex $\displaystyle\int_2^3 \int_1^6 xy\,e^{x+y}\,dy\,dx$

x-bounds y-bounds

$$= \int_2^3 \int_1^6 xe^x \cdot ye^y \,dy\,dx$$

$$= \int_2^3 \left[xe^x \cdot \int_1^6 ye^y\,dy \right] dx$$

by parts!

$$\int_1^6 ye^y\,dy = ye^y\Big|_1^6 - \int_1^6 e^y\,dy$$

$u = y \to du = 1dy$

$dv = e^y dy \to v = e^y$

$$= 6e^6 - 1e^1 - \left(e^y\Big|_1^6 \right)$$

$$= 6e^6 - \cancel{e} - (e^6 - \cancel{e})$$

$$= 6e^6 - e^6 = 5e^6$$

$$= \int_2^3 xe^x \cdot 5e^6 \, dx = 5e^6 \int_2^3 xe^x \, dx$$

Same IBP again.

$$= 5e^6 \left(xe^x \Big|_2^3 - \int_2^3 e^x \, dx \right)$$

$$= 5e^6 \left(3e^3 - 2e^2 - e^3 + e^2 \right)$$

$$= \boxed{5e^6 \left(2e^3 - e^2 \right)}$$ final answer.

Another Example:

$$\int_0^2 \left[\int_0^3 x+y^2 \, dy\right] dx$$

$$= \int_0^2 \left(xy + \frac{1}{3}y^3 \Big|_{y=0}^{3} \right) dx$$

$$= \int_0^2 \left(3x+9\right) dx$$

$$= \frac{3}{2}x^2 + 9x \Big|_{x=0}^{2} = \frac{12}{2} + 18 - (0+0)$$

$$= 6 + 18 = \boxed{24}$$

Final Ex

$$\int_0^1 \left[\int_0^1 3x^2 y^3 \, dx \right] dy$$

$$= \int_0^1 \left(x^3 y^3 \Big|_{x=0}^{1} \right) dy$$

$$= \int_0^1 \left(1^3 y^3 - 0^3 y^3 \right) dy = \int_0^1 y^3 \, dy$$

$$= \frac{y^4}{4} \Big|_{y=0}^{1} = \boxed{\frac{1}{4}}$$