MATH 208, EXAM 2

SECTION 250

Name: Solutions. NUID: ______________________

Instructions.

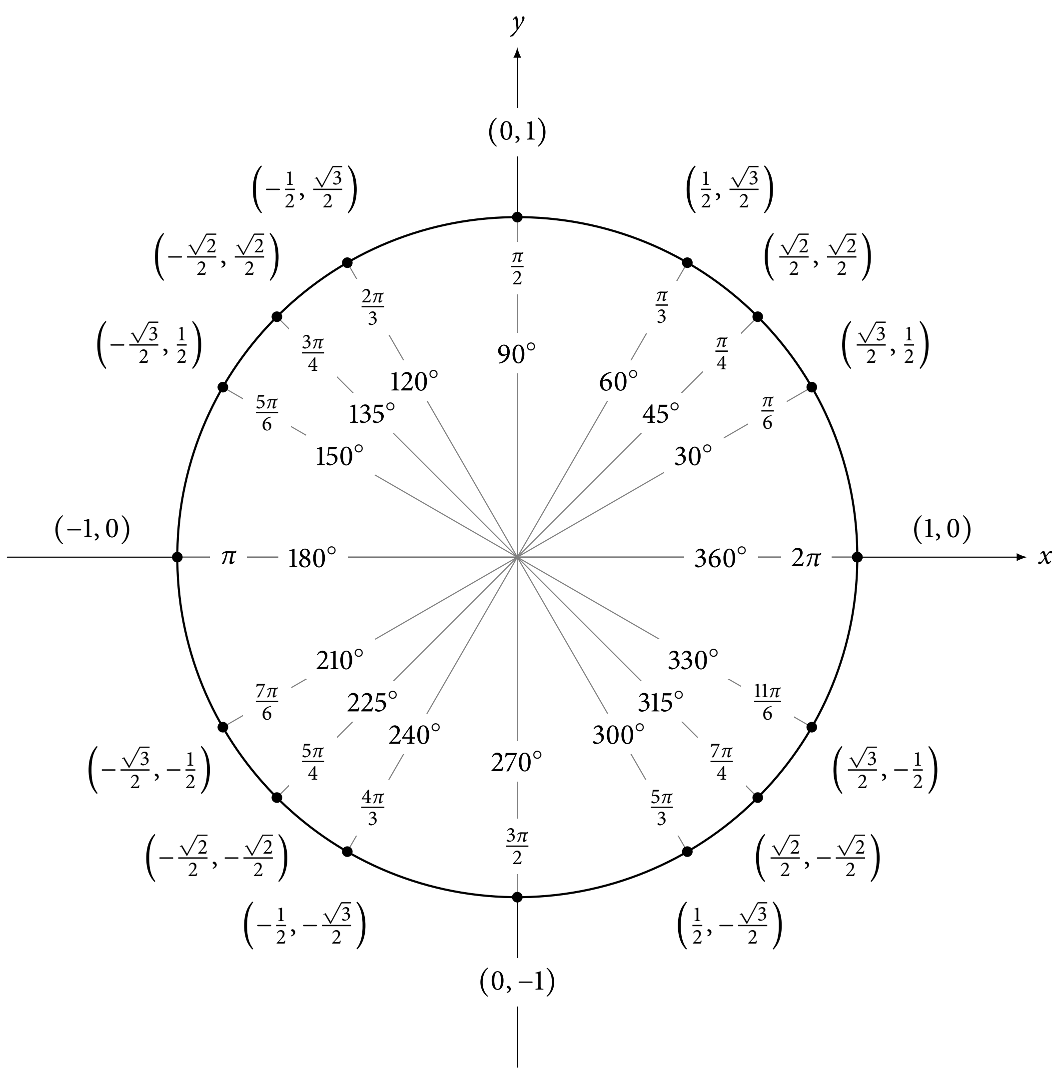
- You should have 8 pages on which 7 problems are printed.
- You have 50 minutes: the exam will begin on the half hour and end promptly, 50 minutes later.
- Show all work unless otherwise specified. What you write on the page must convince me that you understand the problem and its solution.
- Read each problem carefully.
- You do not need to simplify your answers, unless the instructions for a problem indicate otherwise.
- You are not allowed a calculator, notes, textbooks, or access to any electronic devices.
- Don't panic. Good luck!

Date: Fall 2022.

Here are some things you might find useful.

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$dV = \rho^2 \sin\phi \, d\rho d\phi d\theta$$



Problem 1 (8 points). Values of $f(x, y)$ are shown in a table below. Let R be the rectangle $[2, 2.6] \times [5, 5.2]$, i.e., all points (x, y) satisfying $2 \leq x \leq 2.6$ and $5 \leq y \leq 5.2$. Use the table of values to produce a reasonable *underestimate* for the integral $\iint_R f(x, y)\, dA$. (**Do not simplify your answer.**)

	$x = 2$	$x = 2.2$	$x = 2.4$	$x = 2.6$
$y = 5$	4	3	-1	-5
$y = 5.1$	0	2	1	3
$y = 5.2$	-1	-2	-3	-4

Take minimum value in each of 6 0.2×0.1 boxes:

$$0.2 \cdot 0.1 \left(0 + -2 + -1 + -3 + -5 + -4 \right)$$

$$\left(= 0.02 \cdot 15 = 0.3 \right).$$

Problem 2 (8 points). A biologist is studying a population of microorganisms in a circular petri dish of radius 5cm. Her model predicts that the population density of microorganisms should be $\frac{C}{e^r}$ individuals/cm^2 at distance r from the center of the dish, for some constant C. Set up (**but do not solve**) an integral giving the total population of microorganisms in the dish.

$$\boxed{\int_{\theta=0}^{2\pi} \int_{r=0}^{5} \frac{C}{e^r}\, r\, dr\, d\theta}$$

Initials: $\boxed{}$ Math 208, Exam 2

Problem 3 (20 points). Use the method of Lagrange multipliers to find the maximum value taken by the function $f(x,y,z) = 3yz - x^2$ on the plane $\underbrace{-2x + 6y + 6z}_{g(x,y,z)} = 11$.

$\nabla f = \lambda \nabla g$:

$$\left.\begin{aligned} -2x &= f_x = \lambda g_x = \lambda(-2) \\ 3z &= f_y = \lambda g_y = \lambda(6) \\ 3y &= f_z = \lambda g_z = \lambda(6) \end{aligned}\right\} \Rightarrow \begin{aligned} x &= \lambda \\ y &= 2\lambda \\ z &= 2\lambda \end{aligned}$$

Plug into constraint equation!

$$-2(\lambda) + 6(2\lambda) + 6(2\lambda) = 11 \Rightarrow \lambda = \frac{1}{2}$$

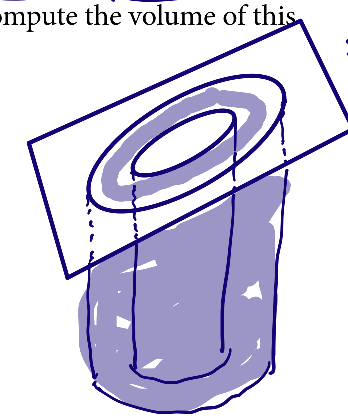
So $x = \frac{1}{2}$, $y = 1$, $z = 1$.

Maximum: $f(\boxed{1/2}, \boxed{1}, \boxed{1}) = \boxed{11/4}$.

Problem 4 (20 points). A thick tube is bounded below by the xy-plane, above by the plane $z = 2x + 5$, and between the two cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 1$. $r=1$

(a) Set up an integral in cylindrical coordinates to compute the volume of this region.

$x = r\cos\theta$ $r=2$



$z=2x+5$

$$\int_{\theta=0}^{2\pi}\int_{r=1}^{2}\int_{z=0}^{2r\cos\theta+5} r\,dz\,dr\,d\theta$$

"correction factor" for cylindrical

(b) Evaluate your integral.

$$= \int_{\theta=0}^{2\pi}\int_{r=1}^{2} r\left(2r\cos\theta + 5 - 0\right)\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[\frac{2r^3}{3}\cos\theta + \frac{5r^2}{2}\right]_{r=1}^{2} d\theta$$

$$= \int_{\theta=0}^{2\pi}\left(\frac{14}{3}\cos\theta + \frac{15}{2}\right) d\theta$$

$$= \left[\frac{14}{3}\sin\theta + \frac{15}{2}\theta\right]_{\theta=0}^{2\pi} = \frac{14}{3}\underbrace{\sin 2\pi}_{=0} + 15\pi - \frac{14}{3}\underbrace{\sin 0}_{=0}$$

$$= \boxed{15\pi}$$

Problem 5 (6 + 6 + 6 points)**.** For each of the following regions W, set up (**but do not evaluate**) an iterated integral in spherical coordinates giving $\iiint_W z\,dV$. (The integrand is the function $f(x,y,z) = z$.)

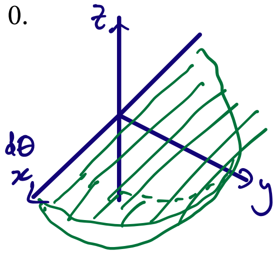
(a) W is the region between the two spheres $x^2+y^2+z^2 = 3$ and $x^2+y^2+z^2 = 16$.

$z = \rho\cos\phi$ $\qquad \rho = \sqrt{3} \qquad \rho = 4$

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=\sqrt{3}}^{4} \rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

(b) W is the set of points (x,y,z) lying between the two spheres $x^2+y^2+z^2 = 3$ and $x^2+y^2+z^2 = 16$ and satisfying $y \geq 0$, $z \leq 0$.

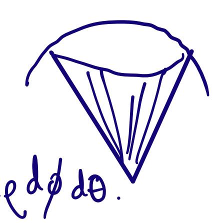
$$\int_{\theta=0}^{\pi}\int_{\phi=\frac{\pi}{2}}^{\pi}\int_{\rho=\sqrt{3}}^{4} \rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$



Below xy-plane:
$\frac{\pi}{2} \leq \phi \leq \pi$

$\phi = \frac{\pi}{4}$

(c) W is the set of points (x,y,z) lying between the cone $\underline{z = \sqrt{x^2+y^2}}$ and the sphere $x^2+y^2+z^2 = 16$. ice cream cone!

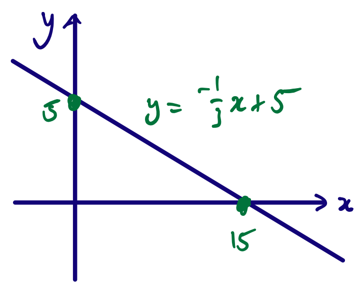


$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{4} \rho\cos\phi\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

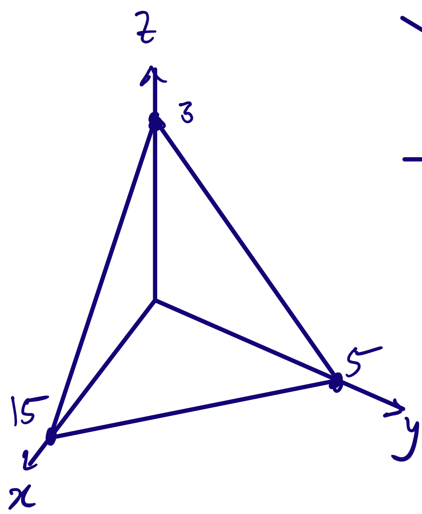
Problem 6 (6 + 6 points)**.** Let T be the tetrahedral region bounded by the xy-plane, the xz-plane, the yz-plane, and the plane $x + 3y + 5z = 15$. Express the integral $\iiint_T f\,dV$ as an iterated integral using the two requested orders of integration.

(a)

$$\int_{x=0}^{15}\int_{y=0}^{-\frac{1}{3}x+15}\int_{z=0}^{\frac{1}{5}(15-x-3y)} f(x,y,z)\,dz\,dy\,dx$$

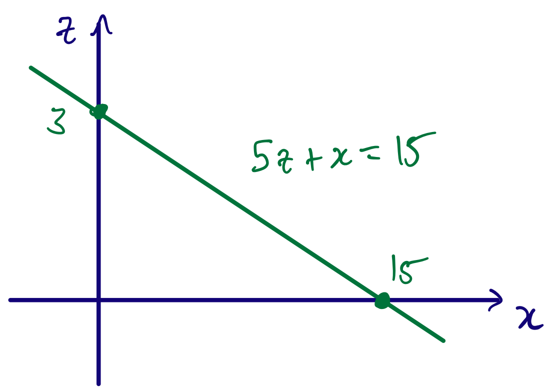


Solve plane equation for appropriate variable



(b)

$$\int_{z=0}^{3}\int_{x=0}^{15-5z}\int_{y=0}^{\frac{1}{3}(15-x-5z)} f(x,y,z)\,dy\,dx\,dz$$



Initials: $\overrightarrow{PQ} = \langle -5,-5,0 \rangle$ Math 208, Exam 2

Problem 7 (7 + 7 points). The two parts are independent.

(a) Let L be the line that is the intersection of the planes $z = 3$ and $x = y$. Notice that the two points $(5, 5, 3)$ and $(0, 0, 3)$ each lie on L. Give a parametrization $\vec{r}(t)$ of L satisfying $\vec{r}(0) = \langle 5, 5, 3\rangle$ and $\vec{r}(2) = \langle 0, 0, 3\rangle$.

(For partial credit, give any correct parametrization of L.)

"Standard" parametrization:

$$\langle 5,5,3\rangle + t\langle -5,-5,0\rangle.$$

Desired parametrization:

$$\vec{r}(t) = \langle 5,5,3\rangle + \frac{t}{2}\langle -5,-5,0\rangle, \quad -\infty < t < \infty$$

(b) Give a vector-valued function that parametrizes the circle of radius 5 centered at $(-3, 4)$ in a **clockwise** direction.

(For partial credit, give any correct parametrization of this circle.)

"Standard" parametrization:

$$\langle -3,4\rangle + 5\langle \cos(t), \sin(t)\rangle, \quad (\text{CCW}) \quad 0 \le t \le 2\pi.$$

To reverse orientation, change t to $-t$:

$$\langle -3,4\rangle + 5\langle \cos(-t), \sin(-t)\rangle, \quad 0 \le t \le 2\pi.$$