12.1 Vector Fields

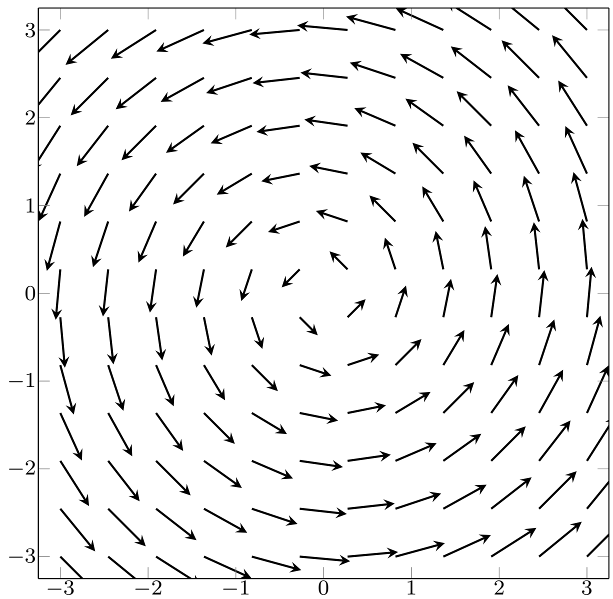
In this section, we will introduce the concept of a vector field, explore how to visualize vector fields, and examine gradient-like vector fields.

Vector Fields

A vector field is a function $\mathbf{F}(\mathbf{x})$ that takes vectors as inputs and returns a vector of the same dimension as an output.

For example, the function $\mathbf{F}(x, y) = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$ is a 2-dimensional vector field.

To visualize a vector field, one draws vectors whose tails are placed at sample points $\mathbf{x}$ and are pointed in the direction of $\mathbf{F}(\mathbf{x})$.



The vector field $\mathbf{F}(x, y) = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$

A common type of vector fields are **gradient vector fields**, which we've already envountered! A gradient vector field is the vector field obtained as $\mathbf{F}(\mathbf{x}) = \boldsymbol{\nabla} f(\mathbf{x})$ where $f(\mathbf{x})$ is a real-valued function. In this case, the function $f(\mathbf{x})$ is called the **potiential function** of the vector field $\mathbf{F}(\mathbf{x})$.

Question 1. Sketch the vector field $\mathbf{F}(x, y) = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ on the domain $[-2, 2] \times [-2, 2]$.

Question 2. Consider the function $f(x, y) = \sqrt{x^2 + y^2}$ on the domain $0 \leq x^2 + y^2 \leq 1$. Sketch the gradient vector field $\mathbf{F}(x, y) = \boldsymbol{\nabla} f(x, y)$ on this domain.

Question 3. Consider the vector field $\mathbf{V}(x, y) = \langle y, -x \rangle$.

(a) Sketch the vector field $\mathbf{V}(x, y)$ on the domain $[-3, 3] \times [-3, 3]$.

(b) Is the vector field $\mathbf{V}(x, y)$ a gradient vector field? If so, find a scalar function $\varphi(x, y)$ such that $\nabla \varphi = \mathbf{V}$. If not, explain why.

Question 4. For each of the following, first, draw a vector field with the given property. Then, write a formula for the vector field you drew

(a) All vectors point towards the point $(1, -1)$.

(b) All vectors are parallel to the x-axis (horizontal) and all vectors on a vertical line have the same magnitude.

(c) All vectors are of length 3 and point towards the origin.