

POLYNOMIAL INVARIANTS OF KNOTS AND LINKS

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ABSTRACT. In this paper, we give a brief overview of knots before diving into knot invariants. We first discuss invariants such as the crossing number and the unknotting number, providing examples of each. Then we introduce three early polynomial invariants, the Alexander-Conway polynomial, the Jones polynomial, and the HOMFLY polynomial. For each polynomial, we give some of the history behind the polynomial, conditions of use on the polynomial, how the polynomial is calculated, and a sample calculation using the figure-eight knot. Finally, we study applications of polynomial invariants, specifically in the proving of the Tait Conjectures.

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1. INTRODUCTION

When observing curves in $\mathbb{R}^3$, we often see curves that loop back around to connect the beginning and end together. Such a closed curve that does not intersect itself on $\mathbb{R}^3$

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is called a knot [1]. Two-dimensional drawings or pictures of knots, called projections, are used to help categorize and observe various knots. Since these projections are on $\mathbb{R}^2$ instead of $\mathbb{R}^3$, knots will often appear to cross themselves at points referred to as crossings. Following along the edge of a knot, we see that at a specific crossing, a section of the knot either goes “over” another section or “under” the other section. However, these crossings are not set in stone, as knots may be deformed and moved. This process of deformation, known as ambient isotopy, means that each knot can have multiple, even infinitely many, projections. As a result, the ability to simplify knots or identify key characteristics of a knot is invaluable for differentiating between knots.

One of the most common ways to simplify a projection is to use Reidemeister moves. We shall call these R1, R2, and R3, respectively. With an R1 move, we simply twist or untwist a strand of the knot, gaining or losing a crossing in the process. If we have a section

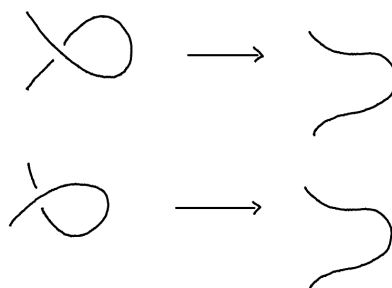


FIGURE 1. Examples of R1 moves.

of the knot K that appears to be “under” another for two or more consecutive crossings, then we can also slide the one on the bottom out from underneath, as shown in Figure 2. This movement, an R2 move, changes the number of crossings by two. For an R3 move, we slide a section of K either over, under, or between two other nearby sections that have a crossing. Note that while R1 and R2 movements change the number of crossings, an R3 move might not change how many crossings we see in a projection, just their relative location. Using these three movements allows us to simplify, or reduce the apparent complexity of, a knot’s

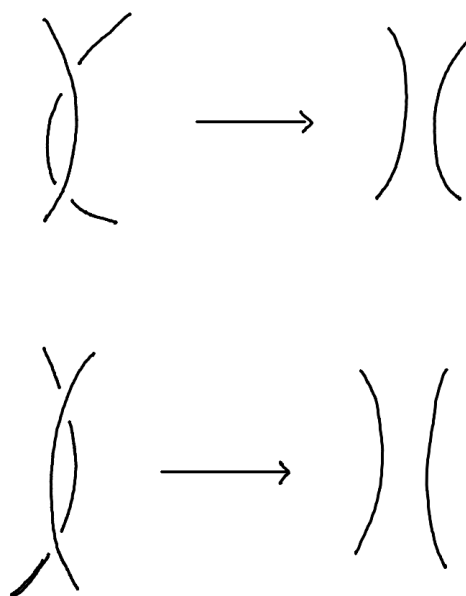


FIGURE 2. Examples of R2 moves.

projection. Should we be able to move a knot to the point of having no crossings left, we have what is called the unknot, or trivial knot.

Essentially, the unknot is a closed loop in the form of a circle for its simplest form. We will also later work with links, which are two or more knots entangled with one another, such that at least one knot must be cut to untangle them.

2. INVARIANTS

In this section, we begin to tackle the issue of distinguishing between different knots. Since we can create an infinite number of projections for the same knot simply by adding an infinite number of R1 or R2 moves, we must look for properties or characteristics of the

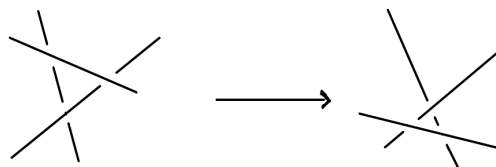


FIGURE 3. An example of a R3 move.

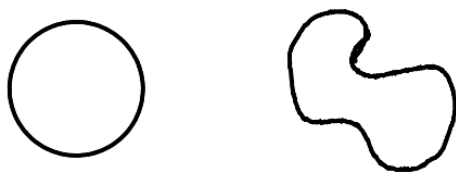


FIGURE 4. Two projections of the unknot.

knot that remain consistent despite any deformations. Specifically, we discuss a collection of properties and characteristics that remain unchanged through all three Reidemeister moves called invariants [1]. When choosing invariants, we lean towards those that are easily computed and provide useful information. For example, an invariant that tells us whether the knot is the unknot gives us helpful information. Some invariants, however, give helpful information but are difficult to compute.

Example 1. One invariant that tells us about the structure of a knot would be the knot's crossing number. The crossing number of a knot, denoted $c(K)$ for some knot K , tells us the minimum number of crossings we could have in a projection of K . Even though we can take a projection K and increase or decrease the number of crossings through Reidemeister moves, there is still an unchanging minimum number of crossings for K . Crossing number lets us distinguish a knot from the unknot, and perhaps even other knots, but this particular invariant still has its own issues. Finding a projection with the minimum number of crossings is incredibly difficult, since there may be just one more move that further reduces it. The discovery of distinct knots of the same crossing number has also prevented this invariant from being the perfect sorting tool for knots. For example, the knots in [Figure 5](#) are distinct but have the same crossing number.

Example 2. Another invariant to consider is a property called **tri-colorability**. Unlike the complex calculations needed for some invariants, all we need is three colors and a projection of the knot in question. Using these three colors, we assign a color to each strand. If we can assign colors such that



FIGURE 5. A drawing of two distinct knots with a crossing number of 5.

- (1) All colors are used, and
- (2) Each crossing has exactly one color or all three colors,

then the knot is **tri-colorable**. The unknot is not tri-colorable since we cannot use all three colors. Similarly, the figure-eight knot, a knot of four crossings, is not tri-colorable either, as it fails (2). A simple knot that fits the conditions to be tri-colorable would be the trefoil knot, a knot of crossing number 3, as shown in [Figure 6](#).



FIGURE 6. The trefoil knot (left) is tri-colorable while the figure-eight knot (right) is not.

Example 3. Our final example of a common invariant would be the **unknotting number** of a knot, denoted $u(K)$. The unknotting number is the minimum number of crossings that would need to be removed for a knot to turn into the unknot [\[3\]](#).

Given that the unknot has no crossings, no crossings must be changed to obtain it, resulting in an unknotting number of 0. However, any knot distinct from the unknot will need at least one crossing changed to become the unknot. This makes a knot's unknotting number useful for distinguishing it from the unknot. Unfortunately, the unknotting number is difficult to calculate and cannot always be found using the simplest projection of the knot.

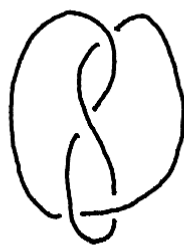


FIGURE 7. The figure-eight knot has an uncrossing number of one.

3. SKEIN RELATIONS AND POLYNOMIAL INVARIANTS

A particular family of invariants that is of interest is a collection of knot polynomials. These knot polynomials are computed from a projection of a knot, but are not affected by the specific projection of that knot used. In other words, these polynomials are not affected by Reidemeister moves and are therefore knot invariants. To fully discuss some of these polynomials, however, we need to define relationships that are called **skein relations**. These skein relations connect a link to the polynomial of links that come from changing a crossing in the projection of the original [1]. Changed crossings, also called **smoothings**, are classified as shown in Figure 8 below. When we refer to these throughout our discussion of polynomial invariants, we will assume that the rest of each oriented link is identical to the others. In other words, the only crossing that differs between L_+ , L_- , and L_0 is the one shown below.

3.1. The Alexander-Conway Polynomial. The first of our polynomials, named after J. W. Alexander in 1928 [2], is a polynomial for oriented links using the variable t . His calculation methods were not only abstract in terms of the knot itself but also involved various concepts from linear algebra. At the time, skein relations had not been defined, but Alexander did

define his polynomial based on an index determined by over and under crossings [2]. Forty-one years later, John Conway found a method of calculating the Alexander polynomial, Δ , of a knot by using just two rules [1].

Rule 1 says that $\Delta(\text{unknot}) = 1$, and is not dependent on what projection of the unknot we look at. Thankfully, this means that once we reach a projection we *know* to be the unknot, we are closer to a stopping point in our calculations. For the second rule, we use the smoothings from Figure 8 above to define our equation as follows.

$$\Delta(L_+) - \Delta(L_-) + \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) \Delta(L_0) = 0$$

Example 4. Let us consider the figure eight knot as was shown in Figure 7 and follow it along a specific orientation. If we take L_+ to be the original knot at one of its over crossings, then L_- will swap that crossing from an over crossing to an under crossing. But then we find that L_- is a projection of the unknot. So $\Delta(L_-) = 1$ by Rule 1. Then, to calculate $\Delta(L_0)$, we smooth out the crossing by cutting both strands and re-attaching them so that the orientation is still followed in both new strands, but those strands no longer cross. Doing so gives us a pair of connected links, so we must once again use the smoothings we have to break them down even further. Repeating this process with $\Delta(L_0) = \Delta(L_{0+})$, gives us $\Delta(L_0) = 1 - t^{\frac{1}{2}} + t^{-\frac{1}{2}}$. Back-substituting this in Rule 2 for the original knot gives us the

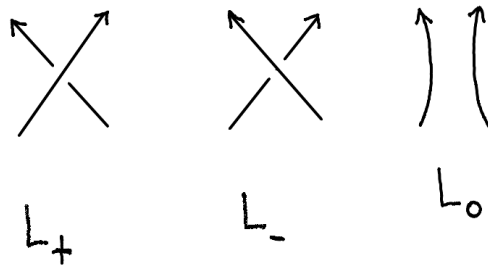


FIGURE 8. The three possible smoothings: L_+ , L_- , and L_0 .

following:

$$\Delta(L_+) - 1 + \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) \left(1 - t^{\frac{1}{2}} + t^{\frac{-1}{2}}\right) = 0$$

Which can be solved for $\Delta(L_+)$ to get

$$\Delta(L_+) = -1 - t^{-1} + t^{-\frac{1}{2}} - t^{\frac{1}{2}} - t,$$

giving us the Alexander-Conway polynomial for the figure-eight knot.

As this polynomial is calculated recursively, more complicated knots will require more steps. However, those calculations can be made more efficient by utilizing a knot's uncrossing number. For example, the figure-eight knot had an uncrossing number of 1, so finding crossing changes that ended in a projection of the unknot took relatively few calculations.

3.2. The Jones Polynomial. Fifteen years after Conway found his method of using skein relations to calculate the Alexander polynomial, mathematician Vaughan Jones discovered a new polynomial for knots. This new polynomial, dubbed the Jones polynomial, is based on a polynomial called the **bracket polynomial**[\[1\]](#). The bracket polynomial uses a total of three rules: one for the unknot, one for the smoothing of over and under crossings, and one for the union of a link with another trivial link. Unfortunately, given how both the second and third rules are defined, an R1 move changes the bracket polynomial. To fix this issue, we require the knot to be oriented. Giving our knot an orientation just means that we must follow a strand of the knot in the same direction until we reach the place we started.

Applying that orientation fix yields the Jones polynomial, denoted by V . Much like the Alexander-Conway Polynomial, we see that Rule 1: $V(\text{unknot}) = 1$. Our second rule is defined by

$$t^{-1}V(L_+) - tV(L_-) + \left(t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)V(L_0) = 0.$$

Notice the similarity between this rule and the second rule for the Alexander-Conway polynomial. Even though they are distinct polynomials, they uphold the same set of smoothings.

Example 5. Now that we have defined the Jones polynomial, we will consider this polynomial for the figure-eight knot. Just as with the Alexander-Conway polynomial, we need to work recursively through crossings until we reach the unknot for each type of crossing change. Going through this process with the figure-eight knot gives the following:

$$V(\text{figure-eight knot}) = t^{\frac{7}{2}} + t^3 - t^{\frac{5}{2}} - t^2 + 1$$

Now, as we look at the Jones polynomial as an invariant, there do exist distinct knots that have the same Jones polynomial. In 1986, an article published by Taizo Kanenobu showed several families of knots that fit this criteria [5]. Unfortunately, this means that the ability to distinguish knots using the Jones polynomial is limited.

3.3. The HOMFLY Polynomial. From the time the Alexander polynomial was first introduced until the discovery of the Jones polynomial, the study of polynomial invariants attracted little attention. After the discovery of the Jones polynomial, however, the search to find other polynomial representations of knots began. Within four months, several independent knot theorists discovered a polynomial invariant in two variables. Taking the last initial of those mathematicians led to this polynomial being named the HOMFLY polynomial [1].

The HOMFLY polynomial, P , is given using the variables m and l , and is defined in two rules just as the other Alexander-Conway and Jones polynomials are. Similar to the other two polynomials, Rule 1 states that $P(\text{unknot}) = 1$. For Rule 2, we will once again reference the smoothings in Figure 7 to define L_+ , L_- , and L_0 . Then

$$lP(L_+) + l^{-1}P(L_-) + mP(L_0) = 0.$$

As the HOMFLY polynomial is also defined using smoothings that change one crossing at a time, we must calculate it recursively, just as we have for the other two polynomials discussed.

Example 6. So then let us once again use the figure-eight knot and calculate its HOMFLY polynomial. Going through a similar set of equations as with the other polynomials, we find that the HOMFLY polynomial of the figure-eight knot is

$$P(\text{figure-eight knot}) = -l^{-1} + ml^{-3} - m^2l^{-2}.$$

The HOMFLY polynomial has been shown to be a generalization of both the Alexander-Conway and the Jones polynomials. In fact, a direct substitution replacing m with $i(t^{-\frac{1}{2}} - t^{\frac{1}{2}})$ and l with it^{-1} gives the Jones polynomial when $i = \sqrt{-1}$. Likewise, substituting $l = i$ and $m = i(t^{\frac{1}{2}} - t^{-\frac{1}{2}})$ give the Alexander-Conway polynomial. This means that the HOMFLY polynomial is capable of conveying the same information as both of those previous polynomials combined and more. However, this does not make the HOMFLY polynomial a perfect invariant, as pairs of distinct knots exist such that both have the same polynomial representation. Thus, the HOMFLY polynomial cannot distinguish all knots.

4. APPLICATION OF KNOT POLYNOMIALS

During the 1870s and 1880s, the Scottish physicist and knot theorist Peter Tait proposed a series of conjectures now known as the **Tait Conjectures** [4]. After the discovery of the above knot polynomials in the 1920s and beyond, modern knot theorists turned their eyes back to the Tait Conjectures with the hope that these newly minted invariants could provide more insight.

The first of these conjectures is in relation to the crossing number of alternating knots: *Any reduced diagram of an alternating link has the fewest possible crossings.* The proof of this conjecture came in 1987 by Louis Kauffman using the Jones polynomial [6]. Kauffman approached this conjecture by first proving the highest and lowest degrees for the Jones polynomial of a generic reduced alternating link using the number of crossings and the separation of regions at crossings. Then he used those degrees to calculate the span of the link and matched it to the number of crossings. In the same timeframe, a similar

Jones polynomial-based proof of this conjecture was published by Kunio Murasugi [7]. Both Kauffman and Murasugi proved additional corollaries in their work, providing a foundation for the proofs of the remaining Tait conjectures.

For example, proven alongside the first conjecture, Kauffman also proved another of Tait's conjectures in 1987: *An achiral alternating link has zero writhe*. This conjecture was also proven in 1988 in a paper by Morwen Thistlewaite that discussed connections between algebraic and geometric interpretations of alternating knots [8].

In the 1990s and beyond, further research on polynomial invariants remained prevalent among knot theorists, even branching into other fields. Applications in various fields of biology and physics continue to be considered and expanded.

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